ON THE GROUPS OF ORDER p^m WHICH CONTAIN OPERATORS OF ORDER p^{m-2}*

BY

G. A. MILLER

BURNSIDE has considered the groups of order p^m (p being any prime) which contain an invariant cyclic subgroup of order p^{m-2}.† Those in which a cyclic subgroup of order p^{m-2} is transformed into itself by an abelian group of order p^{m-1} and of type $(m-2, 1)$ have also been studied.‡ The main object of the present paper is to determine the remaining groups of order p^m ($m > 4$ when p is odd, and $m > 5$ when $p = 2$) which contain a cyclic subgroup of order p^{m-2}. As such a subgroup must be transformed into itself by p^{m-1} operators of the group of order p^m,§ each of these groups which does not come under one of the cases already considered must include the non-abelian group H of order p^{m-1} which contains p cyclic subgroups of order p^{m-2}. The group of isomorphisms (I) of H is of order $p^{m-1}(p-1)$ and contains invariant operators of order p^{m-3} when p is odd and of order p^{m-4} when $p = 2$.‖

Let P_1 and P_2 represent two independent operators of H whose orders are p^{m-2} and p respectively and let $P_1^{p^{m-3}} = P_3$. Suppose also that P_2 has been so chosen that $P_2^{-1}P_1P_2 = P_3P_1$. The group of cogredient isomorphisms (I_2) of H is of order p^2 and of type $(1, 1)$. When p is odd I includes an operator (t_1) of order p such that

$$t_1^{-1}P_1t_1 = P_2P_1, \quad t_1^{-1}P_2t_1 = P_2.$$

Since t_1 permutes the p cyclic subgroups of order p^{m-2} in H cyclically, while some of the operators of I_2 are commutative with each operator of only one of these subgroups, the group generated by I_2 and t_1 is the non-abelian group of

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† BURNSIDE, *Theory of groups of finite order*, 1897, p. 75.

‡ Transactions of the American Mathematical Society, vol. 2 (1901), p. 259.

§ BURNSIDE, Proceedings of the London Mathematical Society, vol. 26 (1895), p. 209. Also, FROBENIUS, Berliner Sitzungsberichte (1895), p. 173.

‖ With respect to the non-cyclic group of order p^2, when p is odd, or p^3, when p is even, all the operators of a division have the same pth power or p^2th power respectively. Cf. Bulletin of the American Mathematical Society, vol. 7 (1901), p. 350; J. W. YOUNG, Transactions of the American Mathematical Society, vol. 3 (1902), p. 189.

order p^3 which contains no operators of order p^2. As this group contains only p of the p^{m-3} invariant operators of I it follows that I contains the non-abelian subgroup of order p^{m-1} which includes no operator of order p^{m-2} but has an invariant operator of order p^{m-3}, whenever p is odd. This subgroup of order p^{m-1} is invariant under I according to Sylow's theorem. It is not difficult to see that the same group is invariant under the group of isomorphisms of the abelian group of type $(m-2, 1)$.

§1. *Determination of the groups when p is even.*

When $p=2$, I is of order 2^{m-1} and its subgroup (I_2) which is composed of the group of cogredient isomorphisms of H is the four-group. It includes an operator t_2 of order 2 such that

$$t_2^{-1} P_1 t_2 = P_1^{-1}, \qquad t_2^{-1} P_2 t_2 = P_3 P_2.$$

This operator is commutative with each operator of I_2 since it is evidently commutative with the operator (t_2') which transforms P_1 into itself and P_2 into $P_3 P_2$. Hence I contains the abelian group of type $(m-4, 1, 1)$ and all the operators of this subgroup transform P_1 into a power of itself. An additional generator of I is t_1 as defined above. It should however be observed that t_1 is commutative with only p^{m-3} operators of H when $p=2$, while it is commutative with p^{m-2} of these operators when p is odd.

It was observed above that I contains an invariant operator of order p^{m-4} when $p=2$. Let t_3 represent the operator of order 2 which is a power of this invariant operator. From the properties mentioned above it follows that

$$t_1^{-1} t_2' t_1 = t_3 t_2', \qquad t_1^{-1} t_2 t_1 = t_2.^*$$

Hence, when $p=2$, I contains a subgroup of type $(m-4, 1)$ which is composed of its invariant operators. It is completely defined by the fact that it contains such a subgroup and two non-commutative operators (t_1, t_2') of order 2 with properties noted above.

We proceed to determine all the groups of order 2^m which contain H and permute its cyclic subgroups of order 2^{m-2}. Such a group must transform H according to a subgroup of order 8 in I, which includes the group of cogredient isomorphisms of H. As all the operators of orders two and four contained in I are included in its subgroup of order 32 there are just four such subgroups of order 8 and each of them is simply isomorphic with the octic group.† They are generated by I_2 and the following four operators of order two respectively:

$$t_1, \quad t_1 t_2, \quad t_1 t_2' t_4, \quad t_1 t_2 t_2' t_4,$$

* These equations may be verified by observing that each member transforms P_1 and P_2 in the same way.

† Cf. PIERPONT, Annals of Mathematics, ser. 2, vol. 1 (1900), p. 140.

where t_4 is an operator of order 4 in the group generated by an operator of order 8 in I.

The group (G_1) generated by H and t_1 contains just 2^{m-4} invariant operators and is conformal with the abelian group of type $(m-2, 1)$; i. e., it contains 2^{a+1} operators of order 2^a $(1 < a < m-1)$ and 7 of order 2. Its four cyclic subgroups of order 2^{m-2} involve, in pairs, the two cyclic subgroups of order 2^{m-3} contained in H. It follows directly from a known theorem that there is no other group which transforms H in the way in which G_1 transforms it.*

The group (G_2) generated by H and $t_1 t_2$ contains only 2 invariant operators. Its operators not contained in H are composed of 2^{m-3} operators of order 2 and 3.2^{m-3} of order 4. Since $P_1^{-2} t_1 t_2 P_1^2 = P_3 P_1^{-4} t_1 t_2$ there can be no other group which transforms H in the same manner as $t_1 t_2$ does. Let G_2' represent the group generated by H and $t_1 t_2' t_4$. Its 2^{m-3} invariant operators are generated by P_1^2 and it is conformal with G_1. As it contains an abelian subgroup of type $(m-2, 1)$ it is not necessary to consider this group here. There is another group (G_3) which transforms H in the same way as G_2' does and contains four cyclic subgroups of order 2^{m-2}. In G_3 all of these contain the same subgroup of order 2^{m-3} while this is not the case in G_2'. Moreover, G_3 contains no operator of order 2 besides those in H and it has no abelian subgroup of type $(m-2, 1)$.

It remains to examine the case when H is transformed in the same way as $t_1 t_2 t_2' t_4$ transforms it. The group (G_4) generated by H and $t_1 t_2 t_2' t_4$ contains only two invariant operators. Besides H it contains 2^{m-2} operators of each of the orders 2 and 8. In the other group (G_5) which transforms H in the same manner as G_4 does, there are 2^{m-2} operators of each of the orders 4 and 8 besides H. There cannot be more than two such groups, since H has only two invariant operators under G_4. Hence *there are just five groups of order 2^m which contain operators of order 2^{m-2} and in which no cyclic subgroup of this order is either invariant or transformed into itself by an abelian group of order 2^{m-1}.* It may be of interest to observe that the group of isomorphisms of H when $p = 2$ is identical with that of the abelian group of type $(m-2, 1)$.

§ 2. *Determination of the groups when p is odd.*

When $p > 2$ the two sets of p conjugate subgroups in H are permuted by I according to an intransitive substitution group of order $p^2(p-1)$, which is obtained by establishing a (p, p) isomorphism between two metacyclic groups of degree p, just as in the case of the abelian group of type $(m-2, 1)$.† The

* Transactions of the American Mathematical Society, vol. 2 (1901), p. 265. The latter part of this theorem clearly assumes that p is odd. It remains true, however, when u_1 is a power of r_1 and the order of r_1 is greater than 4. The general method explained in §2 of the article cited is employed in the present article.

† l. c., p. 261.

groups under consideration must transform the operators of H according to a subgroup of I, which includes I_2, is of order p^3, and permutes the p cyclic subgroups of highest order in H. It is evident that there are just p such subgroups. They are non-abelian and $p-1$ of them include operators of order p^2.

To prove that these $p-1$ subgroups are conjugate under I it seems desirable to employ some additional equations, which we proceed to develop. Let t represent an invariant operator of order p^{m-3} in I and let $t^{p^{m-5}} = t_4$. It may be assumed without loss of generality that $t_4^{-1} P_1 t_4 = P_1^{1+p^{m-4}}$ and $t_4^{-1} P_2 t_4 = P_2$. There are $p(p-1)$ conjugates of $t_1 t_4$ under I. They are

$$t_{\alpha\beta}^{-1}(t_1 t_4) t_{\alpha\beta} \quad (\alpha = 1, 2, \cdots, p-1;\ \beta = 1, 2, \cdots, p)$$

where

$$t_{\alpha\beta}^{-1} P_1 t_{\alpha\beta} = P_1^{\alpha}, \qquad t_{\alpha\beta}^{-1} P_2 t_{\alpha\beta} = P_3^{\beta} P_2.$$

It follows that

$$(A) \qquad (t_{\alpha\beta}^{-1} t_1 t_4 t_{\alpha\beta})^{-1} P_1^{\alpha} (t_{\alpha\beta}^{-1} t_1 t_4 t_{\alpha\beta}) = P_2 P_1^{\alpha + \alpha p^{m-4} + \beta p^{m-3}}.$$

On the other hand

$$(B) \qquad (t_1 t_4)^{-n} P_1^{\alpha} (t_1 t_4)^n = P_2^{\alpha n} P_1^{\alpha(1+p^{m-4})n + \alpha p^{m-3} \alpha(\alpha-1)/2}.$$

The right hand members of (A) and (B) are the same only if

$$n = 1 + \beta p, \qquad \alpha = 1.$$

Hence not more than p of the $p(p-1)$ conjugates of $t_1 t_4$ are powers of $t_1 t_4$, i. e., the operators $t_{\alpha\beta}$ transform $\{t_1 t_4\}$ into at least $p-1$ conjugate groups. It remains to observe that only one of these groups can be in any one (I_3) of the $p-1$ subgroups of order p^3 under consideration.

The last fact follows readily from the isomorphism between I and the given intransitive substitution group of order $p^2(p-1)$. In this isomorphism I_3 corresponds to the subgroup of order p^2 and $\{t_1 t_4\}$ corresponds to an invariant subgroup of order p. The I_3 which includes $t_1 t_4$ can therefore involve only p of the conjugates of $t_1 t_4$ under I. In other words, the conjugates of $t_1 t_4$ are found in $p-1$ conjugates of I_3.

Since these $p-1$ subgroups of order p^3 are conjugate under I it is necessary to consider only two cases, viz.: the one in which H is transformed according to one of these $p-1$ subgroups and the other in which H is transformed by the groups in question according to the subgroup of order p^3 in I, which includes no operator of order p^2. In the former case there are only p^{m-4} invariant operators while each of the groups which belongs to the latter contains p^{m-3} such operators. We proceed to prove that there is only one group (G_1) which comes under the former case, while there are two (G_2, G_2') which come under the

latter. It is not difficult to see that the last one of these groups contains a subgroup of type $(m-2, 1)$.

Let t_5 be an operator of order p^2 which transforms H in the same way as $t_1 t_4$ does and suppose that it has been so chosen that $t_5^2 = P_2$. The group generated by H and t_5 contains no operator of order p besides those of H. That this is the only group in question which transforms H in the same way as G_1 does may be proved in exactly the same manner as the theorem to which reference is made in the last footnote. It may be observed that G_1 is conformal with the abelian group of type $(m-2, 2)$.

The group (G_2) generated by t_1 and H is conformal with the abelian group of type $(m-2, 1, 1)$. In fact, it includes the abelian group of type $(m-3, 1, 1)$ since t_1 is commutative both with P_1^2 and with P_2. The other group G_2', which transforms H in the same manner as G_2 does, may be obtained by the method mentioned in the last footnote. Since it includes the abelian group of type $(m-2, 1)$ it will not be considered here. Hence, *there are two and only two groups of order* p^m $(p > 2$ *and* $m > 5)$ *which include operators of order* p^{m-2} *without containing either an invariant cyclic subgroup of this order or an abelian subgroup of type* $(m-2, 1)$. These two groups are conformal respectively with the abelian groups of type $(m-2, 2)$ and of type $(m-2, 1, 1)$. When $m = 5$ the group G_1 evidently contains an invariant cyclic subgroup of order p^{m-2}; hence there is only one group of order p^5 $(p > 2)$ which contains operators of order p^3 without containing either an invariant cyclic subgroup of this order or the abelian group of type $(3, 1)$.

ON THE CIRCUITS OF PLANE CURVES*

BY

CHARLOTTE ANGAS SCOTT

1. The nature of the individual circuits (or complete branches) that make up a curve of order n has not received very much attention. VON STAUDT (1847) distinguished between odd and even circuits; MOEBIUS (1852, *Ueber die Grundformen der Linien der dritter Ordnung*) by projection on to a sphere from the centre, brought out even more clearly the distinction, since the odd circuit is represented on the sphere by a single line, the even circuit by two distinct lines. CAYLEY (1865, *On Quartic Curves*, Collected Papers, vol. 5, op. 361) applied the Moebius projection to the quartic, and thus proved that not only the individual circuits of any non-singular quartic, but all the circuits at once, can be projected into the finite part of the plane. He pointed out that this conclusion does not hold as regards the non-singular sextic; there exists such a curve, composed of a single circuit, which cannot be projected into the finite part of the plane. On a sphere, this circuit is not confined to one hemisphere. In the concluding paragraph the remark is made that a quartic with one node may consist of two odd circuits; such a quartic is not the projection of any finite curve. CLIFFORD (1870, *Synthetic Proof of Miquel's Theorem*) mentioned another sextic that is not confined to one hemisphere. ZEUTHEN, in his classical discussion of quartic curves (1874, Mathematische Annalen, vol. 7) emphasized the distinction between odd and even circuits, and proved the important theorem that in one, and only one, of the two regions † into which a plane is divided by a non-singular even circuit, odd circuits can lie. He showed (p. 426) that a quartic can be composed of a single even circuit, cutting itself twice, of such a form that every straight line meets it in at least two real points. There may be an additional double point of any kind, or the curve may have a simple circuit, necessarily a simple oval. A quartic of this character, with $p = 0$, can be obtained by quadric inversion from a conic which separates one of the three fundamental points from the other two. If such a circuit be projected on to the sphere, it will be found that the two lines by which it is represented interlace.

2. The theorems mentioned suggest that there is a general theorem as to the existence of circuits that cannot be projected into the finite part of the plane,

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† On a sphere, there are three regions.

that is, circuits some of whose intersections with every straight line are real. Let the maximum number of points in which a circuit is met by a straight line be called its order; let the minimum number be called the index of the circuit; it is obvious that

$$\text{order} - \text{index} \equiv 0 \pmod 2,$$

and that

$$\text{order} - \text{index} \geqq 2.$$

The general theorem can be stated in the form:

For every order n there exist curves, $p = 0$ *or* 1, *formed of a single circuit of index* $n - 2$.

The curves $p = 1$ may have also another circuit, without point- or line-singularities.

Further, for every order n there exist curves, $p = 0$ or 1, formed of a single circuit of index $n - 2r$, where r may have any value from 1 to $n/2$ if n be even, from 1 to $(n-1)/2$ if n be odd.

3. As a preliminary, in order to form some idea of the appearance of such a curve, consider a curve of order n $(p = n - 2)$ composed of $n - 2$ odd circuits through O, namely

$$(x^2 + y^2)\prod_{i=1}^{i=n-2} l_i + \prod_{i=1}^{i=n-2} t_i = 0,$$

where the l's and t's represent real lines through O, alternating. Let these odd circuits be drawn so that they do not all pass through O, as in Fig. 1, for which $n = 5$. If now the first circuit be linked with the second, and the second with the third, as indicated in Fig. 1′, a single circuit of index 3 is formed with the help of two new double points B, C.

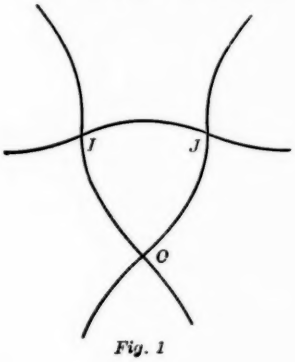

Fig. 1

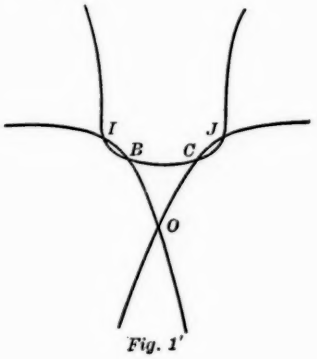

Fig. 1′

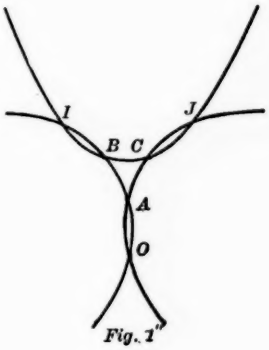

Fig. 1″

Similarly the $n - 2$ circuits, linked by means of $n - 3$ new double points, become a single circuit of index $n - 2$. If this process of deformation is

admissible, the resulting curve is a C_m, $p = 1$, composed of a single circuit, for which

$$\text{order} - \text{index} = 2.$$

A different linking of the circuits is shown in Fig. 1″, accomplished by means of three new double points; the curve is a quintic, $p = 0$.

4. The rigorous proof of the theorem is obtained by means of Cremona transformations. By this means also the validity of this process of linking the circuits is established.

When an oval is subjected to quadric inversion, its position with respect to the fundamental triangle OIJ determines the appearance of the resulting circuit. If the oval does not meet any of the fundamental lines, it transforms into an oval, with or without indentations; if it crosses any of the fundamental lines, but in such a manner as not to enclose any of the fundamental points, it becomes a branch cutting itself, but still such that it can be projected into the finite part of the plane. If however the oval surrounds one of the fundamental points, O, it necessarily meets in at least two real points all conics through OIJ, and since these correspond to the straight lines of the transformed plane, the resulting circuit is of index 2, of the type of the Zeuthen quartic circuit. A similar result follows if the oval surrounds two of the fundamental points.

An oval that passes through one of the fundamental points transforms into an odd circuit, of index 1, passing through the two fundamental points whose correspondents are the two fundamental lines that meet in the point through which the oval passes.

The quintic of which the existence was suggested by the process of deformation can actually be obtained by inversion from a quartic. Let $u = 0$ be a quartic consisting of three ovals; let O, I, J be points on these, chosen for the

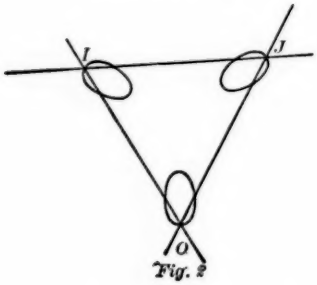

Fig. 2

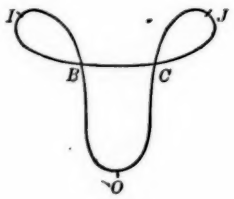

Fig. 2′

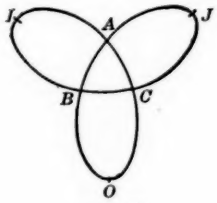

Fig. 2″

sake of simplicity so that the quartic touches a conic $v = 0$ at these points (Fig. 2). The quartics $u + \lambda v^2 = 0$, where λ is a variable parameter, form a system through OIJ, lying outside u if the sign of λ is properly chosen. These invert

into a system of quintics, composed of three odd circuits intersecting at O, I, J. If u, v are chosen so that the whole figure has a single axis of symmetry through O, the two double points B and C (Fig. 2′) present themselves for the same value of λ, and the circuits are linked as in Fig. 1′. If there is triangular symmetry, the three double points A, B, C appear for some one value of λ, (Fig. 2″), and the transformed curve is as shown in Fig. 1″.

In the general Cremona transformation the straight lines of the transformed plane are derived from rational curves passing through the fundamental points in the original plane, forming a Cremona net. If the curve to be transformed entirely surrounds a fundamental point of odd order, it is cut in at least two real points by every curve of the Cremona net; the transformed curve consequently meets every straight line in its plane in at least two real points, and is therefore of index 2. The number of fundamental points inside any one loop, link, or oval, makes no difference, so long as there are fundamental points of odd order outside also. Similarly the presence of a fundamental point of odd order on the boundary of a loop, link, or oval ensures one real intersection other than the fundamental point itself.

5. The simplest type of Cremona net is composed of curves of order q with a fixed $(q-1)$-point and $2(q-1)$ fixed simple points. By means of this net the theorem can be proved. It will be shown in the first place that for every order m there exists a curve with a multiple point of order $m-3$ and $2(m-3)$ links, if $p=1$, but with $2(m-3)+1$ links if $p=0$. Such curves are familiar among quartics, as shown in Fig. 3 ($p=0$) and Fig. 3′ ($p=1$).

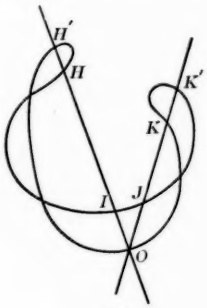

Fig. 3

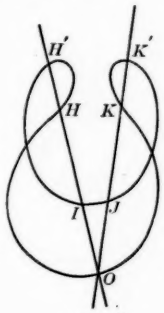

Fig. 3′

A point O is chosen on the part of the curve indicated in the figure, and from O two lines are drawn to meet the final loops of the curve in real points, which is certainly possible for these quartics; these lines meet the quartic again at I, J. By means of a quadric inversion with OIJ as fundamental points the curve is transformed into a quintic, with a double point at O since the line IJ meets

the quartic again in two points. The pairs of points H, H'; K, K' give rise to new double points at I, J, and the resulting quintic is of the desired type. (Figs. 4, 4.′)

Two lines can be drawn through O to meet the loops in real points, H, H', K, K'; these lines meet the base of the curve each in one point, necessarily real, and therefore available as fundamental points I, J. The line IJ meets the quintic again in three points, hence the transformed curve, a sextic,

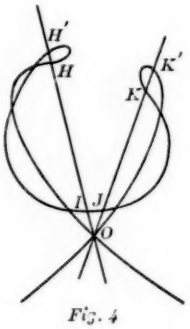

Fig. 4

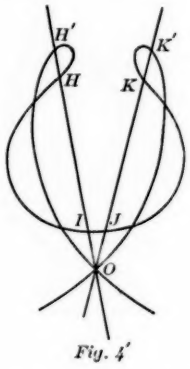

Fig. 4′

has a triple point at O; and as before the points HH', KK' give rise to double points at I, J, and the two series of links in the curve are increased each by unity. The curve is a C_6, with a 3-point, and $2.3 + 1$ links if $p = 0$, 2.3 links if $p = 1$.

In precisely the same manner we obtain from a C_m with an $(m-3)$-point and $2(m-3)+1$ [or $2(m-3)$] links a curve of order $2m-(m-3)-2$, that is, $m+1$, with, at O, a point of order $m-1-1$, that is, $m+1-3$, and $2(m-3)+1+2$, that is $2\{(m+1)-3\}+1$ [or $2(m-3)+2$, that is, $2\{(m+1)-3\}$] links. It is not necessary to consider whether the branches that form the multiple point at O are all real; this is immaterial. What is essential is that the links be all real, and this is shown by the manner in which the curve is produced.

6. We now apply a Cremona transformation of the kind mentioned, choosing q so that there shall be a fundamental point available for the interior of every link. This requires $q = m-2$. The curve C_m with an $(m-3)$-point and $2(m-3)$ links is transformed by means of curves of order $m-2$ with a fixed $(m-3)$-point and $2(m-3)$ fixed simple points. The transformed curve is therefore of order

$$n = m(m-2) - (m-3)^2 = 4m - 9,$$

and the index, to which every link contributes 2, is at least $2 \times 2(m-3)$.

Since however $4(m-3) = 4m-9-3$, which gives for *order − index* the odd value 3, the index must be greater than $4(m-3)$. It cannot be greater than $4m-9-2$, hence

$$\text{index} = 4m-11 = n-2.$$

This proves the theorem for every n of the form $4m-9$, that is, $4k-1$; for the intermediate values all that is necessary is to place 1, 2, 3 of the fundamental points on the boundaries of links instead of inside. Every such placing of a fundamental point diminishes both order and index by unity, and thus leaves undisturbed the relation, now proved for all values of n,

$$\text{order} - \text{index} = 2.$$

The fundamental curves for the Cremona net here used are

(i) $2(m-3)$ straight lines joining O to the other fundamental points, singly;

(ii) a C_{m-3} with an $(m-4)$-point at O, passing singly through the other fundamental points.

Since each of the straight lines (i) meets C_m in 3 points that do not lie at the fixed points, the C_n has $2(m-3)$ triple points if $n = 4m-9$, but 1, 2, 3 of these are replaced by double points if $n = 4m-9$ diminished by 1, 2, 3. The fundamental C_{m-3} gives rise to a multiple point of order $4(m-3)$ on C_n, diminished by 1, 2, 3 for the special cases named; the order of this point is therefore in all cases $n-3$. There are also the unaltered double points of the original curve. These make up exactly the right number for the C_n with $p = 0$ or 1.

If now h of the fundamental points are placed outside the links instead of inside, the order is not affected, but the index is diminished by $2h$. In this way, for the curve of order $4m-9$ the index can be made as low as 1, instead of $4m-11$. For the intervening orders the lowest index, 1 or 0, is obtained by placing the last 1, 2, 3 fundamental points on the base of the curve.

This completes the proof of the theorem.

7. For greater values of p it can be shown that the index can attain to certain specified values, but proof is still lacking that these are the highest possible values * for any given n and p. If p be of the form $4t-2, 4t-1, 4t, 4t+1$, then for every order n, where $n \geqq 2(t+1)$, there exists a curve composed of a single circuit for which

$$\text{order} - \text{index} = 2(t+1).$$

For the proof of this, the first step is to demonstrate the existence of curves of every order m of the type shown in Figs. 5 and 5′, each with an $(m-3)$-point at O. For any such curve

* In fact, they are not the highest values, as witness the sextic, $n=6$, $p=10$, index $=2$.

$$p = \tfrac{1}{2}(m-1)(m-2) - \tfrac{1}{2}(m-3)(m-4)$$

diminished or not by unity; that is, $p = 2m - 6$, Fig. 5, and $p = 2m - 5$, Fig. 5′.

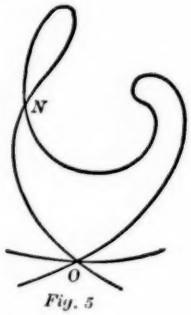

Fig. 5

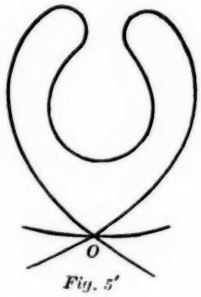

Fig. 5′

If the existence of such a curve, $u_m = 0$, be assumed for any value of m, it follows at once for the value $m + 2$.

(1) Let m be even, the desired C_{m+2} is given by

$$f_2 u_m = \lambda F_m \phi_2,$$

with a sufficiently small value for λ, where f_2 is an imaginary line-pair through O, F_m a set of m imaginary lines through O, and ϕ_2 a pair of imaginary lines, not through O, but intersecting at N if this other node exists.

(2) Let m be odd; then with the same interpretation for f, F, and ϕ the conditions are fulfilled by

$$f_2 u_m = \lambda F_{m-1} z \phi_2,$$

where $z = 0$ is the straight line at infinity.

This proves that if curves of the kind indicated exist for order m, they exist for order $m + 2$. They do exist for $m = 4$; and the equation

$$y u_4 = \lambda F_2 z \phi_2$$

proves that they do exist for $m = 5$, and consequently for every value of m. Any such curve may have also certainly one oval or even two, possibly more.

If we now transform these curves by quadric inversion, exactly as for the case of $p = 0$ or 1, we obtain from each a series of curves, as shown in Fig. 6, where curves of the same order are represented in the columns, and curves with the same p in the rows.*

*These diagrams show the nodes and links accurately as to number and general situation; they are not intended to show directions to infinity or inflexions. In fact, some inflexions not actually existing on the curves have been inserted on the links, for clearness in drawing.

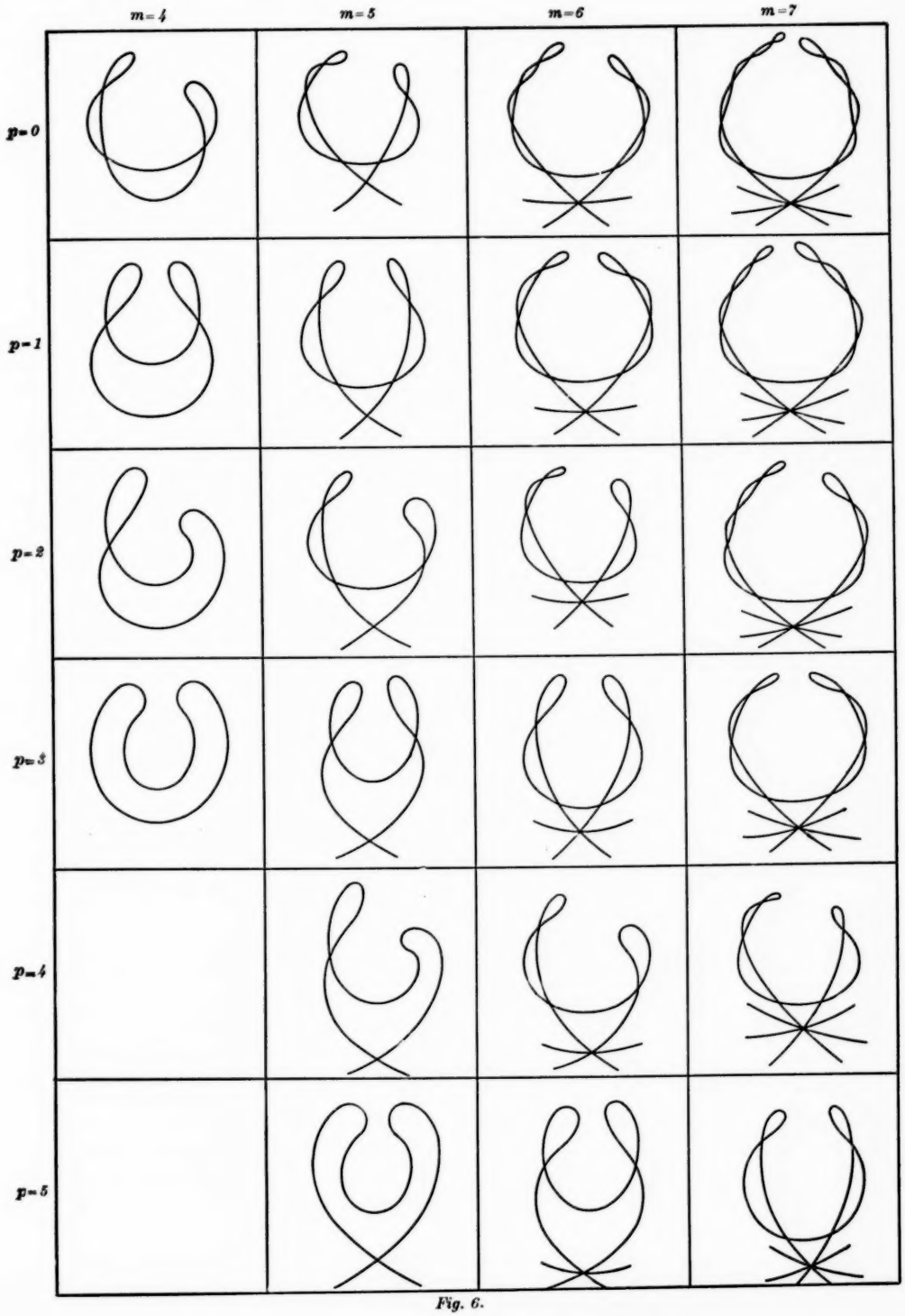

Fig. 6.

Any such curve is a C_m with an $(m-3)$-point and q links, where $q = 2m-5-p$. This is to be subjected to a Cremona transformation by means of curves of order $m-k$, with a fixed $(m-k-1)$-point and $2(m-k-1)$ simple points, where $2(m-k-1) \leqq 2m-5-p$, in order that there may be a fundamental point available for every link. Hence $2k \leqq p+3$. The fundamental points left over, in number $p+3-2k$, can be placed on the base of the C_m; their effect is then to diminish the order of the transformed curve, for which we have the value

$$n = m(m-k)-(m-3)(m-k-1)-(p+3-2k) = 4m-k-p-6.$$

Exactly as before,

$$\text{index} \geqq 2(2m-5-p),$$

and therefore

$$\text{index} \geqq n-(p+4-k).$$

The index is therefore greatest, for a given n and p, when k is as great as possible, that is, when

$$2k = p+3 \quad \text{or} \quad p+2.$$

(1) If p be even, $2k = p+2$;

$$\text{index} \geqq n-\tfrac{1}{2}(2p+8-p-2) \geqq n-\tfrac{1}{2}(p+6);$$

hence

$$\text{order}-\text{index} \leqq \tfrac{1}{2}(p+6).$$

If $\frac{1}{2}(p+6)$ be even, that is, if $p = 4t-2$,

$$\text{order}-\text{index} = 2(t+1).$$

If $\frac{1}{2}(p+6)$ be odd, that is, if $p = 4t$,

$$\text{order}-\text{index} \leqq 2t+3,$$

hence

$$\text{order}-\text{index} = 2(t+1).$$

(2) Similarly if p be odd, it is either of the form $4t-1$ or of the form $4t+1$, and in both cases,

$$\text{order}-\text{index} = 2(t+1).$$

Smaller values for the index, with or without a corresponding diminution in the order, can of course be obtained as before by placing some of the fundamental points on the boundaries of links, or outside the links, instead of inside.

8. In the construction of one of these complicated circuits by the linking of simpler circuits the Zeuthen circuit, derived by inversion from a simple oval, presents itself as fundamental, ranking with the simple oval and the simplest

type of odd circuit. Two nodes are essential, and four real inflexions (Fig. 7), unless there is a cusp or another node (Fig. 7′), which obviates the necessity for two of the four inflexions. For this circuit the name "double-odd circuit" has been suggested.* Topologically, it is derivable by deformation from two odd circuits, the six real inflexions being reduced to the four actually existing by means of the extra node due to the linking of the circuits.

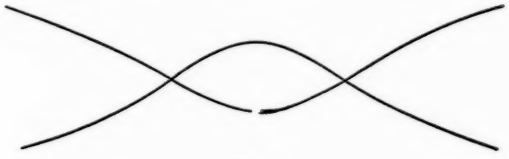

Fig. 7

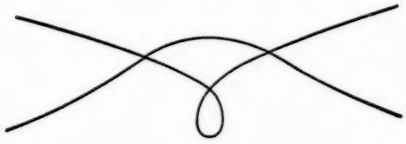

Fig. 7′

If $u = 0$ be one of the curves of order m with an $(m-3)$-point and q links, of which k are occupied by fundamental points in order to give rise to a circuit of index $2k$ or $2k+1$, then $u = \lambda$, where λ is a small constant of proper sign, contains at least q ovals, of which k are occupied by fundamental points. By the Cremona transformation each of these k ovals gives rise to a double-odd circuit, and the single complicated circuit, $u = 0$, arises from the linking of these k double-odd circuits with the circuit derived from the base of the curve. Thus the circuit of index $2k+1$ is derived from k double-odd circuits and one other; or, we may say, it is derived (topologically) from $2k+1$ odd circuits. When $1, 2, 3$ fundamental points are placed on boundaries, $1, 2, 3$ of the double-odd circuits are replaced by simple odd circuits. In any case, the index of any curve here considered is equal to the number of the odd circuits from which the curve can be derived, topologically, by the process of linking.

Since each odd circuit has three real inflexions, a fact proved by MOEBIUS, the circuit of index i, produced from i odd circuits with the help of $i-1$ new double points if $p = 1$, has at least $3i - 2(i-1)$, that is, $i+2$ real inflexions. In particular, the curve for which $i = n-2$ has at least n real inflexions if $p = 1$, and at least $n-2$ if $p = 0$. KLEIN'S equation connecting the numbers of the real singularities, namely,

* R. GENTRY, 1896, *On the Forms of Plane Quartic Curves.*

order + no. real inflexions + 2 no. isolated tangents
= class + no. real cusps + 2 no. isolated points,

becomes in this case (since there cannot be an isolated tangent, which would require four imaginary intersections with a straight line)

$$\text{order} + \text{no. real inflexions} = \text{class}$$
$$= 2n, \text{ if } p = 1,$$
$$= 2n - 2, \text{ if } p = 0.$$

Hence

$$\text{no. real inflexions} = n, \text{ if } p = 1,$$
$$= n - 2, \text{ if } p = 0.$$

If however $p = 0$ in virtue of an isolated point instead of an extra node on the circuit itself, the number of real inflexions $= n$.

9. Any odd circuit in the transformed plane meets the fundamental straight lines each in an odd number of points, and therefore corresponds to a circuit passing an odd number of times through each of the simple fundamental points in the original plane. Now this circuit, passing in and out of the links, has at least $2k$ real intersections with the C_m. Hence not only every straight line, but also *every odd circuit*, cuts the transformed curve in $n - 2$ (in $n - 2r$) real points. This indicates an essential difference between the circuits here considered and the non-singular sextic circuit of index 2, since, as has already been mentioned, there are odd circuits that do not meet this sextic. It seems therefore that it may be necessary to introduce some such term as *circulation*, to denote the number of odd circuits from which a given one can be derived by deformation, or, perhaps more conveniently, the minimum number of times that the circuit must be crossed in passing from an arbitrary point on the sphere to the opposite point, when the path is entirely at our disposal. This aspect of the case I hope to consider at some future time; it is sufficient at present to remark that obviously

$$\text{index} - \text{circulation} \equiv 0 \pmod 2,$$

and that for the complicated circuits here considered

$$\text{index} = \text{circulation}.$$

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NOTE ON THE

REAL INFLEXIONS OF PLANE CURVES*

BY

CHARLOTTE ANGAS SCOTT

The determination of the possible situation of the real inflexions of a curve $u = 0$ is often practically very difficult. The work can sometimes be simplified by means of the fact that real inflexions lie only in the region for which $u_{11}u_{22} - u_{12}^2$ is zero or negative, where u_{11}, u_{22}, u_{12} are written for $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, $\partial^2 u/\partial x \partial y$. This expression, G, depends on the relation of the curve $u = 0$ to the line $z = 0$; a corresponding expression can of course be obtained with reference to any line, and for a real inflexion every such expression must either be zero or have a negative value.

The proof is extremely simple. The polar conic of any point is

$$x^2 u_{11} + y^2 u_{22} + z^2 u_{33} + 2yzu_{23} + 2zxu_{31} + 2xyu_{12} = 0;$$

for a double point or a point of inflexion this represents a pair of straight lines, namely, the tangents at the double point, or the inflexional tangent together with another line which does not pass through the point of inflexion. This line-pair will be imaginary if any one such function as $u_{11}u_{22} - u_{12}^2$ is positive; it will be real if no one of these functions is positive, that is, if every one is either zero or negative. This is ensured if any one such function is negative, or if two are zero.

The expression G is regularly used in discriminating between real and imaginary tangents at a node, but it does not appear to have been used in considering the possibility of real inflexions. If we equate it to zero, we obtain a curve $G = 0$, derived from u and the line z, the locus of points whose polar conics meet z in coincident points. If z is the line at infinity, G is the locus of points whose polar conics are parabolas, including parallel straight lines and coincident straight lines, in which last case the points are cusps or higher singularities.† The curve G divides the plane into regions; the polar conics of points in the

* Presented to the Society December 28, 1901. Received for publication August 16, 1902.

† The higher singularities here referred to are double points with coincident tangents; multiple points of order higher than 2 lie on $G = 0$ for a different reason, namely, on account of the vanishing of u_{11}, u_{22}, etc.

region in which the expression G is positive meet the line z in imaginary points (ellipses and imaginary straight lines) and the polar conics of points in the region defined by G negative meet z in real points (hyperbolas and real straight lines).

It would seem that the curve thus derived from u and a line, as the locus of points whose polar conics touch the line, or as the envelope of the so-called second polars of points on the line, merits consideration. The name "diacritic" suggests itself; G is the diacritic (or diacritic curve) of z with respect to u. If u be a cubic, G is simply the "Poloconik" of the line.

The diacritic of z is the same with respect to all curves of the family $u = k$ ($u = kz^n$ in homogeneous coördinates); it separates the region in which isolated points can lie from the region in which the other nodes (and all real inflexions) are to be found, and thus it must plainly be of importance in the consideration of the critic centres of this pencil. These critic centres are the points common to the first polars of points at infinity, $u_1 + \lambda u_2 = 0$; for this linear system they take the place of an envelope for a more general system, while the diacritic is the envelope of the second polars of points at infinity, $u_{11} + 2\lambda u_{12} + \lambda^2 u_{22} = 0$.

The diacritic is the locus of points of contact of curves $u_1 =$ constant with curves $u_2 =$ constant. This fact is also expressed in the more general statement that the increment of u_1 along the curve $u_2 = 0$ depends at any point on the sign of G (since $\delta u_1 = u_{11}\delta x + u_{12}\delta y$, where $u_{12}\delta x + u_{22}\delta y = \delta u_2 = 0$), and consequently changes sign only when $G = 0$. This shows that in passing along $u_2 = 0$ from one critic centre to the next we pass over G; in general the critic centres are alternately isolated points and nodes with real tangents.

Bryn Mawr College, Pennsylvania.

LA THÉORIE DES PLAQUES ÉLASTIQUES PLANES*

PAR

JACQUES HADAMARD

L'équilibre des plaques élastiques planes et, plus particulièrement, la mise en équation de ce problème, ont fait l'objet de travaux bien connus tels que ceux de LAGRANGE, de POISSON, de KIRCHHOFF, de M. BOUSSINESQ, de M. MAURICE LÉVY, etc. Non seulement les méthodes employées par ces différents auteurs sont très diverses, mais les résultats eux-mêmes n'ont pas toujours été concordants.

Il semble d'ailleurs que les méthodes auxquelles nous venons de faire allusion puissent, tout au moins lorsqu'on admet, comme on doit le faire, l'existence de pressions extérieures exercées sur les faces de la plaque, être aujourd'hui améliorées dans le sens de la rigueur mathématique; autrement dit dans le sens indiqué par M. Hilbert dans une communication récente.† C'est ce que j'ai tenté de faire dans ce qui va suivre. Je n'y ai d'ailleurs que fort incomplètement réussi; cet insuccès tient peut-être, nous le verrons, à ce que les problèmes physiques les plus simples peuvent être en relation avec des questions mathématiques difficiles et non encore abordées.

I.

Le problème qui nous occupe est, en effet, au moins théoriquement, l'un des plus simples et de ceux dont la mise en équation semble devoir offrir le moins de difficultés. Il n'est autre chose qu'un cas particulier d'un problème connu, puisqu'une plaque n'est qu'un corps solide de forme particulière.

Cette circonstance doit nous faire abandonner toute une catégorie de méthodes qui ont été quelquefois proposées pour arriver à notre objet actuel. Je veux parler de celles qui consistent à traiter la question directement, sans faire intervenir les équations générales de l'élasticité, en partant d'hypothèses plus ou moins analogues à celles qui ont servi à établir ces équations. Il est clair que de telles hypothèses sont nécessairement des conséquences des premières à moins qu'elles ne soient en contradiction avec elles, et que, dans les deux cas, elles doivent être rejetées. Nous aurons, au contraire, à utiliser les équations générales de l'élas-

* Presented to the Society at the Ithaca meeting, October 26, 1901. Received for publication March 31, 1902.

† Congrès des Mathématiciens, Paris, 1900.

ticité en faisant des hypothèses spéciales — hypothèses toujours vérifiées dans les conditions où le phénomène se présente expérimentalement* — sur les *données* du problème: autrement dit, sur la forme du corps étudié (celui-ci devant avoir une dimension très petite par rapport aux deux autres et être, en tous ses points, très voisin du plan des xy) et sur la nature des forces qui lui sont appliquées.

Les hypothèses que nous ferons à ce dernier point de vue consisteront a èxclure certaines distributions de forces exceptionelles, autrement dit, présentant certaines relations particulières d'égalité approchée (l'ordre d'approximation étant celui de l'épaisseur 2ϵ de la plaque).

Ces hypothèses seront enoncées en leur lieu. Mais nous serons obligés de leur en adjoindre une autre qui, bien qu'ayant été adoptée comme évidente par tous les auteurs qui ont traité de ces questions, ne se justifie pas par les mêmes raisons que celles dont nous venons de parler, et par conséquent, devra être rejetée dans une théorie définitive.

Cette hypothèse concerne la différentiation des quantités qui vont intervenir dans les calculs. Les quantités en question dépendent des trois coordonnées x, y, z dont les deux premières, ainsi que nous l'avons dit, déterminent la projection du point sur le plan horizontal (le plan de la plaque) pendant que la troisième correspond au déplacement en épaisseur. Supposons qu'une des quantités etudiées (déplacements, tensions, etc.) soit, dans toute l'étendue de la plaque, d'un certain ordre de petitesse. Nous admettrons que cet ordre est aussi celui des quantités obtenues en différentiant la première par rapport à x ou à y.

Il est clair qu'un tel fait n'est nullement évident à priori. Si la quantité U est, en valeur absolue, plus petite qu'un certain nombre P, en tout point de la plaque, la formule des accroissements finis montre bien que sur tout segment parallèle a l'axe des x et dont la longueur n'est pas très petite, il existe des points dans lesquels $\partial U/\partial x$ est inférieur, en valeur absolue, à une quantité assignable et dans un rapport fini avec P. Mais rien ne prouve que ces points ne soient pas séparés par d'autres en lesquels cette relation de grandeur cesse d'avoir lieu. C'est ce qui arriverait si la quantité U était soumise à des oscillations très rapides; par exemple, si U était de la forme

$$U = A \sin \frac{x}{\epsilon}.$$

Ce fait, qui n'est d'ailleurs point relatif aux données de la question, mais aux propriétés de la solution cherchée, devrait donc, en tout rigueur, être démontré. Nous l'admettrons néanmoins dans ce qui va suivre.

Par contre, la même hypothèse ne sera pas faite en ce qui concerne la différentiation par rapport à z. Nous ne pouvons pas, en effet, refaire pour cette

* C'est ainsi qu'il n'est pas légitime de supposer nulles les pressions exercées sur les faces de a plaque.

dernière le raisonnement qui vient de nous servir pour les dérivations relatives à x ou à y. Une quantité étant de l'ordre de P dans toute notre plaque, il se pourrait fort bien que sa dérivée, par rapport à z fut *partout* de l'ordre de P/ϵ.

De même, nous n'admettrons l'hypothèse précédente que dans les parties de la plaque suffisament éloignées des bords. Pour les points dont la distance à ceux-ci est de l'ordre de ϵ, nous ne la considérerons plus comme légitime.

D'après ce qui précède, nous admettrons que les dérivées des projections u, v, w, du déplacement sont (sauf près des bords) du même ordre de grandeur que ces projections elles-mêmes. Ce fait est, au reste, considéré comme évident dans tous les travaux, qui traitent de l'élasticité: nous ne l'accepterons pas ici pour les dérivées par rapport à z.

II.

Les équations que nous écrirons ne sont autres, au fond, que celles qui interviennent dans l'exposition donnée de cette théorie par de SAINT VENANT.* Elles n'en diffèrent que par une évaluation plus rigoureuse des ordres de grandeur. Nous nous conformerons donc aux notations de l'ouvrage en question, et nous désignerons par

$$\mathfrak{g}_x = \frac{\partial u}{\partial x}, \quad \mathfrak{g}_y = \frac{\partial v}{\partial y}, \cdots; \qquad \mathfrak{g}_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \cdots, \cdots,$$

les composantes de la déformation: par t_{xx}, t_{xy}, $\cdots$ celles de l'effort; par A, B, C, les composantes de la force agissante par unité de masse; par 2ϵ l'épaisseur de la plaque. Les coefficients d'élasticité seront supposés tous du même ordre de grandeur.† Soit k une quantité de cet ordre de grandeur.

Soit kP le maximum des tensions t_{xx}, t_{xy}, etc. En vertu de notre hypothèse de différentiation, les derivées $\partial t_{xx}/\partial x$, $\cdots$, $\partial t_{yy}/\partial y$ seront, au plus (en tout point suffisament éloigné du bord) de l'ordre de grandeur de kP. Pour abréger l'écriture, nous désignerons encore par P toute quantité dont le rapport à P n'est pas très grand. Dans ce système de notation, par conséquent, on pourra écrire

$$\frac{\partial t_{xx}}{\partial x} = kP, \qquad \frac{\partial t_{yy}}{\partial y} = kP, \qquad \cdots. \tag{1}$$

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* CLEBSCH, *Théorie de l'élasticité des corps solides.* Traduction de SAINT VENANT et FLAMANT, ch. 5, pages 689–733 (Note du § 73); 1883.

† Plus exactement, soit k_1 le plus grand des coefficients d'élasticité, c'est à dire des coefficients par lesquels il faut multiplier les composantes de la déformation pour obtenir les tensions t_{xx}, $\cdots$. Inversement, exprimons les composantes $\mathfrak{g}_x$, $\cdots$ en fonction des tensions: soit l_2 le plus grand coefficient qui figure dans les équations linéaires ainsi obtenues. Nous supposerons que le produit $k_1 l_2$ (qui est un pur nombre) n'est pas un nombre très grand; qu'en particulier, le produit $k_1 l_2 \epsilon$ est une des quantités que nous regardons comme du même ordre de grandeur que ϵ.

Prenons alors la première équation de l'équilibre interne

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} + A = 0.$$

Multiplions la par dz et, ainsi que le fait de SAINT VENANT, intégrons de $-\epsilon$ à z (où z est compris entre $-\epsilon$ et $+\epsilon$). Le résultat est manifestement, d'après les relations (1), de la forme

$$t_{xz} - (t_{xz})_{-\epsilon} + \int_{-\epsilon}^{z} A dz = k\epsilon P. \tag{2}$$

En particulier, pour $z = +\epsilon$, il vient

$$(t_{xz})_{+\epsilon} - (t_{xz})_{-\epsilon} + \int_{-\epsilon}^{+\epsilon} A dz = k\epsilon P.$$

Or, les valeurs des tensions t_{xz} sont supposées données sur les deux faces de la plaque. Si ces quantités, ainsi que l'expression

$$\int_{-\epsilon}^{+\epsilon} A dz,$$

n'étaient pas toutes de l'ordre de $k\epsilon P$, il faudrait qu'il y eût, entre les pressions exercées sur les deux faces et les forces agissant sur la masse, des relations particulières comme il s'en produirait, par exemple, si l'on agissait à l'endroit considéré sur les deux faces avec les deux branches d'un étau ; ou encore si, la plaque étant soumise à son propre poids, on exerçait sur elle une pression qui neutralise à peu près exactement ce poids. Ce sont précisément de telles relations que nous nous sommes réservé, en commençant, le droit d'exclure. Nous devons donc admettre que dans la somme

$$(t_{xz})_{+\epsilon} - (t_{xz})_{-\epsilon} - \int_{-\epsilon}^{+\epsilon} A dz$$

chacun des termes est (abstraction faite des bords dont nous parlerons plus loin) au plus du même ordre de grandeur que la plus grande valeur prise par la somme entière, autrement dit, que t_{xz} est, sur les deux faces et, par conséquent (en vertu de la relation (2)) dans toute la masse, de l'ordre de $k\epsilon P$. De plus il en est de même de l'intégrale

$$\int_{-\epsilon}^{+\epsilon} A dz$$

et comme la force A est, dans la pratique, toujours sensiblement constante, on doit admettre que cette force est dans un rapport fini avec kP.

Pareillement, t_{yz} est de la forme $k\epsilon P$ et B de l'ordre de kP.

Prenons maintenant la troisième équation de l'équilibre interne

$$\frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} + C = 0.$$

Si nous admettons, conformément a notre hypothèse générale, que $\partial t_{xz}/\partial x$, $\partial t_{yz}/\partial y$ sont du même ordre que t_{xz}, t_{yz}, cette équation, intégrée par rapport à z, nous donnera, comme tout à l'heure,

$$(t_{zz})_{+\epsilon} - (t_{zz})_{-\epsilon} + \int_{-\epsilon}^{+\epsilon} C dz = k\epsilon^2 P.$$

Ici encore, à moins de données exceptionelles pour lesquelles, en effet, les conclusions se trouveraient modifiées, nous avons le droit, d'après cette équation, de faire l'hypothèse que t_{zz} est partout du même ordre que $k\epsilon^2 P$.

Ainsi les tensions t_{xz}, t_{yz} sont très petites par rapport à t_{xx}, t_{xy}, t_{zy}, et la tension t_{zz} est très petite par rapport aux premières. On voit que cette conclusion, que de SAINT VENANT justifie en préjugeant à priori de l'allure des phénomènes,* est démontrable en toute rigueur, du moins une fois admise notre hypothèse de différentiation. Toutefois, les conclusions auxquelles nous venons de parvenir comportent une restriction essentielle: *elles n'ont plus lieu dans le voisinage du bord de la plaque.* Cela tient, d'une part à ce que nous abandonnons dans cette région, l'hypothèse de différentiation, utilisée par nous dans le raisonnement qui précède; d'autre part, à ce que les hypothèses faites tout à l'heure sur la distribution des pressions données sur les faces ne sont plus, ici, vérifiées dans les conditions ordinaires des expériences. Ce sont, en effet, des actions *combinées* (telles que celles des *deux* branches d'un étau) qu'on exerce souvent sur les contours des plaques, de sorte que la concordance particulière dont nous avons exclu la possibilité dans les parties médianes se présente, au contraire, aux bords. Nous n'admettrons donc nullement, dans le voisinage de ceux-ci, que t_{zx}, t_{zy}, t_{zz} soient de la forme $k\epsilon P$ ou $k\epsilon^2 P$.

Quant à la quantité C, elle devra être au plus de l'ordre de $k\epsilon P$. Au reste,

* Par exemple, l'auteur, après avoir indiqué (loc. cit., p. 697) les intégrations par rapport à z que nous lui empruntons dans le texte, ajoute "on voit que les composantes t_{zz} aux t_{xy} n'ont de valeurs, à l'intérieur d'un tronçon de plaque, que celles qu'elles peuvent avoir sur une des deux bases, plus ce qui vient des forces A, B agissant sur sa masse. Ces forces locales n'ont qu'une influence insignifiante qui n'est presque rien en comparaison de ce qui vient à la fois de toutes les forces agissant sur le reste de la plaque ainsi que sur ses bords, par les réactions des appuis ou autrement, et dont les effets accumulés se transmettent au tronçon à travers ses quatre faces latérales, ce qui s'applique surtout aux composantes agissant horizontalement."

Au reste, le défaut d'une évaluation rigoureuse pour les ordres de grandeur se fait encore sentir dès le troisième alinéa de la page 697, dont il parait malaisé de préciser le sens. Il semblerait, en effet, résulter de cet alinéa que les derivées des tensions t_{zz}, t_{xy}, t_{yy} par rapport à x et à y sont négligeables *devant ces tensions elles-mêmes:* assertion manifestement inexacte, qui n'a pu entrer dans la pensée de l'auteur.

dans la plupart des cas usuels, les forces A, B, C sont de beaucoup inférieures aux limites mêmes que nous venons de leur assigner et n'ont qu'une influence négligeable.

III.

Les coefficients d'élasticité étant supposés d'un même ordre de grandeur k, les équations

$$t_{xx},\quad t_{xy},\quad t_{yy} = kP, \tag{3}$$

$$t_{xz},\ t_{yz} = k\epsilon P;\qquad t_{zz} = k\epsilon^2 P, \tag{4}$$

montrent évidemment que les six composantes de déformation sont de l'ordre de P. De plus, si l'on résout les trois dernières d'entre elles par rapport à g_{zx}, g_{zy}, ϑ_z en y remplaçant les seconds membres par zéro, on ne fera sur les valeurs de ces inconnues que des erreurs de l'ordre de ϵP.

Or les équations que l'on obtiendrait ainsi sont

$$\frac{\partial W}{\partial g_{zx}} = 0, \qquad \frac{\partial W}{\partial g_{zy}} = 0, \qquad \frac{\partial W}{\partial \vartheta_z} = 0. \tag{5}$$

W étant l'énergie interne spécifique.

Soit $W_1(\vartheta_x, g_{xy}, \vartheta_y)$, ce que devient W lorsqu'on y met pour g_{zx}, g_{zy}, ϑ_z leurs valeurs tirées des équations (5). On a, moyennant les mêmes équations,

$$\frac{\partial W}{\partial \vartheta_x} = \frac{\partial W_1}{\partial \vartheta_x}, \quad \frac{\partial W}{\partial g_{xy}} = \frac{\partial W_1}{\partial g_{xy}}, \quad \frac{\partial W}{\partial \vartheta_y} = \frac{\partial W_1}{\partial \vartheta_y}.$$

Dès lors (puisque nous ferions sur g_{xz}, g_{yz}, ϑ_z, en écrivant les équations (5), des erreurs de l'ordre de ϵP), les expressions de t_{xx}, t_{xy}, t_{yy} seront de la forme

$$t_{xx} = \frac{\partial W_1}{\partial \vartheta_x} + k\epsilon P, \quad t_{xy} = \frac{\partial W_1}{\partial g_{xy}} + k\epsilon P, \quad t_{yy} = \frac{\partial W_1}{\partial \vartheta_y} + k\epsilon P. \tag{6}$$

Considérons maintenant la dérivée $\partial^2 w/\partial z\,\partial x$. En vertu de notre hypothèse de différentiation, cette quantité sera du même ordre que $\partial w/\partial z$. c'est à dire d'ordre P.

Soient alors u_0, v_0, w_0 les valeurs de u, v, w au feuillet moyen (l'ordonnée z étant comptée à partir de ce feuillet). On aura

$$\frac{\partial w}{\partial x} = \frac{\partial w_0}{\partial x} + \epsilon P.$$

Mais l'équation

$$g_{zx} = \frac{\partial z}{\partial u} + \frac{\partial w}{\partial x} = P$$

donnera, dans ces conditions

$$u = u_0 - z\frac{\partial w_0}{\partial x} + \epsilon P, \tag{7}$$

et l'on aura, de même

$$v = v_0 - z\frac{\partial w_0}{\partial y} + \epsilon P, \tag{7'}$$

d'où

$$\begin{aligned} t_{xx} &= t^0_{xx} - zt'_{xx} + k\epsilon P, \\ t_{xy} &= t^0_{xy} - zt'_{xy} + k\epsilon P, \\ t_{yy} &= t^0_{yy} - zt'_{yy} + k\epsilon P, \end{aligned} \tag{8}$$

$t'_{xx}, t'_{xy}, t'_{yy}$ étant ce que deviennent t_{xx}, t_{xy}, t_{yy} en vertu des équations (6) et au terme $k\epsilon P$ près quand on y remplace u par $\partial w_0/\partial x$ et v par $\partial w_0/\partial y$, soit, symboliquement

$$\begin{aligned} t'_{xx} &= \frac{\partial W_1}{\partial \vartheta_x}\left(\frac{\partial^2}{\partial x^2}, \quad 2\frac{\partial^2}{\partial x \partial y}, \quad \frac{\partial^2}{\partial y^2}\right) w_0, \\ t'_{xy} &= \frac{\partial W_1}{\partial g_{xy}}\left(\frac{\partial^2}{\partial x^2}, \quad 2\frac{\partial^2}{\partial x \partial y}, \quad \frac{\partial^2}{\partial y^2}\right) w_0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{9}$$

Dès lors le calcul fait au no. 3 (pages 694 à 696) de la Note citée de SAINT VENANT conduit à l'équation

$$2\frac{\epsilon^3}{3} W_1\left(\frac{\partial^2}{\partial x^2}, \quad 2\frac{\partial^2}{\partial x \partial y}, \quad \frac{\partial^2}{\partial y^2}\right) w_0 = f(x, y) + k\epsilon^3 P, \tag{10}$$

où $f(x, y)$ a l'expression donnée par de SAINT VENANT (no. 3, équation (f)). De même, on a tous les éléments nécessaires pour effectuer le calcul présenté au no. 7 de la même Note ; et l'on aura des équations tout analogues à celles obtenues en cet endroit, (équations (z), (z')), mais qui devront être complétées par des termes de la forme $k\epsilon^2 P$.

IV.

Avant de chercher à évaluer la quantité P, nous nous occuperons de former les conditions aux limites, question qui a été, comme on le sait, la source des plus grandes difficultés soulevées par la théorie des plaques. La solution de ces difficultés doit, comme on le sait également, être cherchée dans l'état particulier des bords de la plaque, lesquels éprouvent des déformations autres que les régions intérieures. Un tel fait s'introduit tout naturellement d'après ce qui précède, puisque nous renonçons à admettre, pour ces bords, les hypothèses qui ont servi de base à nos raisonnements précédents.

Nous tracerons donc, à une petite distance (de l'ordre de ϵ) du bord de la plaque,* un cylindre vertical C. A l'intérieur de ce cylindre, les conclusions

* Les conclusions que nous allons obtenir subsistent dans ce qu'ils ont d'essentiel, si la distance en question est moindre que ϵ^{1-a}, a étant un nombre positif suffisament petit.

précédemment obtenues seront valables. Au contraire, dans la région annulaire (que nous appelerons la *marge*) extérieure à C nous ne saurons plus qu'une chose, c'est que les diverses tensions t_{xx}, t_{xy}, t_{yy}, t_{xz}, t_{yz}, t_{zz}, sont au plus, de l'ordre de kP.

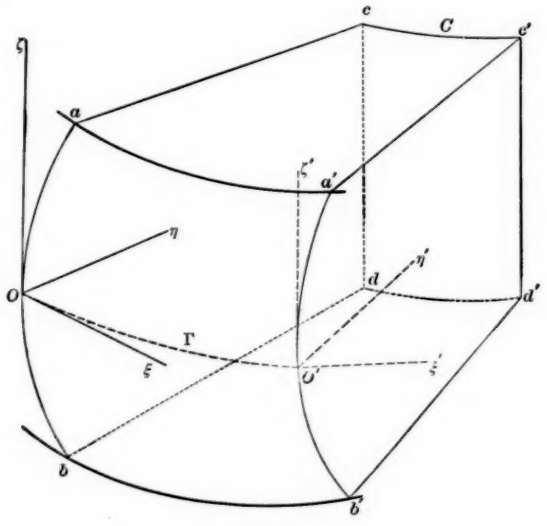

FIG. 1.

Menons, normalement au contour Γ de la plaque en un point quelconque de celui-ci, un plan qui coupera le cylindre C suivant une génératrice cd et la marge suivant le polygone mixtiligne $abcd$ (voir la fig. 1). Soient O un point de la section ainsi obtenue (par exemple comme sur la figure, un point situé sur le contour); $O\xi$, $O\eta$, $O\zeta$, trois axes, dirigés l'un suivant la tangente au contour Γ, le second, suivant la normale horizontale à cette courbe, le troisième, suivant la verticale. L'ensemble des forces élastiques qui s'exercent sur un des cotés de la section $abcd$ aura, par rapport aux axes $O\xi$, $O\eta$, $O\zeta$, trois projections que nous désignerons par X, Y, Z, trois moments désignés par μ_s, μ_n, μ_z, les ordres de grandeur de ces diverses quantités étant évidemment au plus

$$X,\ Y,\ Z = k\epsilon^2 P, \qquad \mu_s,\ \mu_n,\ \mu_z = k\epsilon^3 P. \tag{11}$$

Ainsi qu'il a été dit plus haut, l'hypothèse relative à la différentiation est conservée pour les différentiations le long du contour. Donc les quantités dX/ds (ds étant l'élément d'arc de Γ), dY/ds, dZ/ds sont encore de l'ordre de $k\epsilon^2 P$; les quantités $d\mu_s/ds$, $d\mu_n/ds$, $d\mu_z/ds$, de l'ordre de $k\epsilon^3 P$.

La section $abcd$ et une section voisine $a'b'c'd'$, séparée de la première par un arc ds du contour, intercepteront entre elles une portion de la tranche verticale

de la plaque, et l'ensemble des pressions extérieures exercées sur cette petite surface aura, par rapport aux axes $O\xi$, $O\eta$, $O\zeta$, trois moments que nous désignerons, conformément aux notations de l'ouvrage cité, par $M_s ds$, $M_n ds$, 0 : * et trois projections que nous désignerons par $S' ds$, $N' ds$, $W' ds$.

Ecrivons l'équilibre du solide $abcda'b'c'd'$. Celui-ci est soumis :

1°. Aux forces extérieures données.

2°. Aux forces élastiques développées sur la face $abcd$.

3°. Aux forces élastiques développées sur la face $a'b'c'd'$. Si, par rapport à cette face, on mène les axes $O'\xi'$, $O'\eta'$, $O'\zeta'$ analogues à $O\xi$, $O\eta$, $O\zeta$, les forces élastiques en question auront pour projections et pour moments par rapport à ces axes $X + dX$, $Y + dY$, $Z + dZ$, $\mu_s + d\mu_s$, $\mu_n + d\mu_n$, $\mu_z + d\mu_z$. On en déduit, par un calcul élémentaire, en tenant compte de l'angle $d\theta = ds/\rho$ que fait $O'\xi'$ avec $O\xi$, les projections et les moments de ces mêmes forces par rapport aux axes $O\xi$, $O\eta$, $O\zeta$.

4°. Les forces élastiques qui s'exercent suivant la face $cdc'd'$. Ces dernières s'évaluent à l'aide des formules précédemment écrites, comme il est expliqué dans la note citée de SAINT VENANT (nos. 13–18).

L'équation des moments par rapport à $O\zeta$ ne nous fournit aucun renseignement, car, les quantités qui y figurent, et dont les unes sont à conserver dans le résultat, les autres à éliminer, sont toutes du même ordre de grandeur que $k\epsilon^2 P$.

Les équations des projections horizontales nous donnent les formules (o_1), (o_0') (no. 14) de SAINT VENANT complétées par des termes de la forme $k\epsilon^2 P$ soit, lorsque la plaque est isotrope,

$$k\epsilon^2 P + 2\epsilon\left\{\left[a_1\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) - 2f\frac{\partial v_0}{\partial y}\right]\cos p + f\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\sin p\right\} = U' = -S'\sin p + N'\cos p,$$

(12)

$$k\epsilon^2 P + 2\epsilon\left\{\left[a_1\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) - 2f\frac{\partial u_0}{\partial x}\right]\sin p + f\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\cos p\right\} = V' = S'\cos p + N'\sin p,$$

p étant l'angle de la normale $O\eta$ avec l'axe des x.

Dans l'équation des moments de flexion, s'introduisent les moments μ_s et μ_n (ainsi que leurs derivées par rapport à s). Ces moments sont de la forme $k\epsilon^3 P$ et ne font que se joindre aux termes du même ordre provenant des seconds membres des équations (8), de sorte qu'il vient (comparer SAINT VENANT, équation (t_1), no. 16, p. 713),

$$(13) \qquad -\frac{2\epsilon^3}{3}(t'_{xx}\cos^2 p + 2t'_{xy}\sin p\cos p + t'_{yy}\sin^2 p) = M_s + k\epsilon^3 P.$$

* On sait que le moment par rapport à $O\zeta$ est de l'ordre de ds^2 et non de ds.

Prenons, au contraire, les équations des projections verticales et des moments de torsion: nous aurons

$$W' = \frac{\partial Z}{\partial s} - \frac{2\epsilon^3}{3}\left[\cos p\left(\frac{\partial t'_{xx}}{\partial x} + \frac{\partial t'_{xy}}{\partial y}\right) + \sin p\left(\frac{\partial t'_{xy}}{\partial x} + \frac{\partial t'_{yy}}{\partial y}\right)\right] + Q + k\epsilon^3 P,$$

(14)

$$M_n = -Z + \frac{d\mu_n}{ds} - \frac{\mu_s}{\rho} + \frac{2\epsilon^3}{3}\frac{\partial}{\partial s}\left[\sin p \cos p\,(t'_{xx} - t'_{yy}) + (\cos^2 p - \sin^2 p)\,t'_{xy}\right] + k\epsilon^3 P,$$

Q étant une quantité connue, fonction des efforts donnés, et qui figure dans l'équation (d_2) (page 716) de SAINT VENANT.

Les moments μ_s et μ'_n sont, nous l'avons dit, de l'ordre de $k\epsilon^3 P$. Mais il n'est pas de même de Z. Si donc nous ne voulons conserver (outre les termes qui nous intéressent) que des quantités de l'ordre de $k\epsilon^3 P$, il faut s'éliminer Z entre les deux équations précédentes, c'est à dire former la combinaison de KIRCHHOFF.

Il y a lieu d'observer que l'existence d'efforts verticaux considérables exercés sur les bords de la plaque ne suffirait pas, à elle seule, à introduire notre quantité Z. S'il en était ainsi, en effet, Z serait encore négligeable (à savoir de l'ordre de $k\epsilon^3 P$), en tous les points du contour où les efforts verticaux donnés seraient nuls ou du même ordre que dans l'intérieur. Or cela n'a pas lieu: c'est ce dont il est facile de s'assurer dans des cas particuliers (tels que celui de la plaque circulaire) et c'est ce qui résulte à priori de ce que pour Z négligeable, on aurait à écrire séparément les deux équations (14), ce qui donnerait trois conditions aux limites, au lieu de deux, sur certaines parties du contour.

Il est donc nécessaire, non seulement d'admettre éventuellement l'existence d'efforts autres que ceux qui peuvent avoir lieu dans l'intérieur, mais encore d'abandonner, près des bords, l'hypothèse de différentiation.

V.

Pour évaluer l'ordre de grandeur de P, on peut partir d'une hypothèse classique dans cette théorie.

On sait qu'un solide dont une dimension est très petite peut être déformé de manière que les composantes de la déformation en chaque point soient très petites par rapport aux déplacements (même aux déplacements relatifs) de ces points, le rapport de ces composantes à ces déplacements étant de l'ordre de la dimension très petite.

Convenons d'appeler *flexion* toute déformation qui satisfait à la condition précédente.

L'hypothèse dont il s'agit consiste à admettre que la déformation cherchée est une flexion.

Dans ce cas, la quantité que nous avons designée par P sera très petite par rapport au plus grand des déplacements u, v, w cherchés : elle sera comparable au produit de ce déplacement par ϵ.

Dès lors, la question peut être considérée commé résolue. Car la quantité $k\epsilon^3 P$ pourra être négligée, comme étant d'ordre supérieur au premier membre, dans l'équation (10). De même, les quantités de la forme $k\epsilon^3 P$ par lesquelles nous avions complété les équations (z), (z') pourront être négligées, et ces équations regardées comme exactes.

De plus, si nous supposons que les tensions conservent encore le même ordre de grandeur dans le voisinage des bords, nous pourrons faire abstraction des termes $k\epsilon^2 P$ des équations (12) et $k\epsilon^3 P$ des équations (13), (14).

L'hypothèse ainsi utilisée est bien, contrairement à ce qui parait au premier abord, une hypothèse sur les données du problème et que nous avons le droit de faire.

Supposons, en effet, les efforts F qui agissent sur la plaque decomposés en deux parties F', F'', dont la première, agissant seule, produirait une flexion, la seconde non. Supposons de plus que les efforts F' ne soient pas très petits par rapport aux F''. Alors les composantes de déformation produites en chaque point par F' ne seront pas très petites par rapport à celles qui sont dues à F''. Or ceci exige, d'après ce que nous supposons sur F' et F'', que les *déplacements* produits par F' soient très grands par rapport à ceux qui correspondent à F''. La déformation sensible à l'oeil sera donc une flexion, sauf dans le cas tout exceptionnel où les efforts capables de produire une flexion seraient infiniment petits par rapport aux autres.

Dans ce cas, les résultats pourraient être fort différents : la déformation produite serait évidemment beaucoup plus petite pour la même intensité d'effort que dans les circonstances habituelles.

Le fait que la déformation obtenue est une flexion est donc bien une hypothèse sur les données du problème, puisqu'il peut avoir lieu, ou non, suivant la nature de ces données ; et c'est bien une hypothèse légitime puisqu'elle n'est en défaut que dans des cas exceptionnels (et qu'il serait même assez difficile de réaliser expérimentalement).

Toutefois cette manière d'opérer soulève quelques critiques. D'abord, le raisonnement précédent n'est valable que si l'on est certain qu'il existe des flexions, c'est à dire que leur existence est compatible non seulement (comme il est bien connu) avec les données de la géometrie, mais encore avec celles de la Mécanique, avec les équations de l'équilibre élastique.

Ce point résulte (du moins en l'absence de forces agissant sur les éléments de masse) des résultats de M. MAURICE LEVY.* Celui-ci a écrit les équations d'une infinité de déformations satisfaisant aux équations de l'élasticité et qui

*Journal de Mathématiques, 3 Série, t. 3, 1877.

sont des flexions. On peut d'ailleurs établir directement l'existence de déformations de cette espèce en partant des flexions dont l'existence a été établie au point de vue géométrique. Il suffira de calculer les forces X, Y, Z qu'il est nécessaire de faire agir sur l'unité de masse du corps pour le maintenir en équilibre dans ces conditions. Ces quantités sont de l'ordre des forces élastiques correspondant aux déformations considérées, c'est à dire du produit de $k\epsilon$ par le déplacement. Si alors nous considérons un corps solide de dimensions finies en tous sens et dont fasse partie la plaque donnée, et que nous fassions agir sur ses différents points des forces coïncidant avec X, Y, Z (ou, plus généralement, avec $X+A$, $Y+B$, $Z+C$) dans l'intérieur de notre plaque et qui soient du même ordre partout ailleurs, il est clair que les déplacements et les déformations produites dans ces nouvelles conditions seront du même ordre que les composantes de déformation primitives et, par conséquent, très petits par rapport aux déplacements primitifs. En les combinant avec ceux-ci on aura une déformation possédant la propriété cherchée.

Mais on peut faire à la méthode précédente un autre reproche. D'après ce que nous avons dit plus haut, sa légitimité résulte de ce que, pour une même intensité de forces agissantes, les déformations qui possèdent le caractère de flexion sont beaucoup plus intenses que les autres. Mais ce fait, lui aussi, est une conséquence nécessaire, mathématique, des données générales de l'élasticité. Il devrait donc pouvoir être mis en évidence par nos calculs. Or, ce n'est point ce qui a lieu dans la théorie que nous venons d'exposer. *

VI.

On peut arriver à un résultat plus satisfaisant si l'on emploie un théorème, non demontré, il est vrai, mais que les progrès accomplis dans ces dernières années permettront sans doute d'établir aisément.

Nous avons vu que u_0, v_0 sont liés aux forces données, d'une part, par deux équations aux derivées partielles analogues aux équations (z), (z') de SAINT VENANT, d'autre part, par les conditions aux limites (12). Soient donc les équations aux derivées partielles †

$$
(15)\qquad
\begin{aligned}
\mathfrak{a}_1 \frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)-\mathfrak{f}\frac{\partial}{\partial y}\left(\frac{\partial u_0}{\partial y}-\frac{\partial v_0}{\partial x}\right)&=\alpha,\\
\mathfrak{a}_1 \frac{\partial}{\partial y}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)-\mathfrak{f}\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial y}-\frac{\partial v_0}{\partial x}\right)&=\beta,
\end{aligned}
$$

* Au reste, cette seule circonstance qu'il suffise de montrer l'existence des flexions, sans qu'il soit besoin de rien prouver sur le degré de généralité de ces sortes de déformations, met en évidence le caractère artificiel de cette manière d'opérer.

† Nous prenons, pour simplifier, les équations relatives à la plaque isotrope.

dans lesquelles les fonctions inconnues u_0, v_0 doivent satisfaire aux conditions aux limites

$$(16)\quad \begin{cases} \left[a_1\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)-2f\frac{\partial v_0}{\partial y}\right]\cos p+f\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)\sin p=\gamma, \\ \left[a_1\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)-2f\frac{\partial u_0}{\partial x}\right]\sin p+f\left(\frac{\partial u_0}{\delta y}+\frac{\partial v_0}{\partial x}\right)\cos p=\delta. \end{cases}$$

Supposons qu'on ait établi :

1°. que le problème ainsi posé a une solution, et une seule ;

2°. que les valeurs de u_0, v_0 et de leurs dérivées en tout point appartenant au domaine considéré sont limitées en fonction de α, β, γ, δ, et de leurs dérivées.

S'il en est ainsi, les composantes $\mathfrak{S}_x$, $\mathfrak{S}_y$, g_{xy} sont limitées en fonction de α, β, γ, δ (et de leurs derivées) en chaque point de feuillet moyen. En se reportant aux équations (7) (7′), on voit immédiatement que partout, ces composantes sont limitées en fonction de α, β, γ, δ, ϵw, ϵP. Mais comme les composantes restantes g_{xz}, g_{yz}, $\mathfrak{S}_z$ sont liées aux premières par les relations (4), on voit que P, qui est comparable à la plus grande valeur des composantes de la déformation, est également limité en fonction de α, β, γ, δ, et de ϵw.

Ici, l'on a

$$\alpha = k\epsilon P-\frac{1}{2\epsilon}\left[(t_{zx})_{+\epsilon}-(t_{zx})_{-\epsilon}+\int_{-\epsilon}^{+\epsilon}A\,dz\right],$$

$$\beta = k\epsilon P-\frac{1}{2\epsilon}\left[(t_{zy})_{+\epsilon}-(t_{zy})_{-\epsilon}+\int_{-\epsilon}^{+\epsilon}B\,dz\right],$$

$$\gamma = \frac{U'}{2\epsilon}-k\epsilon P,$$

$$\delta = \frac{V'}{2\epsilon}-k\epsilon P.$$

Donc P est au plus l'ordre de la plus grande des quantités*

$$\frac{U'}{2\epsilon},\quad \frac{V'}{2\epsilon},\quad \frac{(t_{zx})_{+\epsilon}-(t_{zx})_{-\epsilon}+\int_{-\epsilon}^{\epsilon}A\,dz}{2\epsilon},$$

$$\frac{(t_{zy})_{+\epsilon}-(t_{zy})_{-\epsilon}+\int_{-\epsilon}^{\epsilon}B\,dz}{2\epsilon},\qquad \text{et } \epsilon w.$$

* Nous admettons, pour simplifier, que les dérivées de ces quantités sont du même ordre qu'elles-mêmes.

Or, d'autre part, les équations où figure w, savoir les équations (10), (13), (14), nous font connaître cette quantité (ou plutôt certaines de ses combinaisons différentielles) par une somme de termes P et de termes dans lesquels les trois quantités connues

$$f(x, y), \qquad M_s, \qquad W' - A + \frac{dM_n}{ds},$$

que l'on peut appeler les *efforts verticaux*, sont divisées par ϵ^3. Donc l'une au moins des inconnues w, P est au moins de l'ordre des quotients ainsi obtenus.

Nous allons faire l'hypothèse, évidemment légitime, que les efforts verticaux ne sont pas très petits par rapport aux efforts horizontaux

$$U', \quad V', \quad (t_{zx})_{+\epsilon} - (t_{zx})_{-\epsilon} + \int_{-\epsilon}^{\epsilon} A dz, \quad (t_{zy})_{+\epsilon} + (t_{zy})_{-\epsilon} + \int_{-\epsilon}^{\epsilon} B dz.$$

Alors ce sera certainement w qui sera comparable au quotient des efforts verticaux par ϵ^3. Quant à P, il sera de l'ordre de ϵw. Enfin les quantités $\alpha, \beta, \gamma, \delta$ seront de l'ordre de $\epsilon^2 w$ et il en sera de même des déplacements horizontaux *du feuillet moyen*, savoir de u_0, v_0.

Au contraire, les déplacements horizontaux des points suffisament éloignés du feuillet moyen seront de l'ordre de ϵw.

Ces conclusions sont bien celles qu'il s'agissait d'établir et la solution est ainsi achevée.

Ainsi qu'il a été dit plus haut, le lemme de calcul intégral dont je viens de me servir n'a pas été démontré à ma connaissance. Mais les équations aux dérivées partielles (19) ont été étudiées par M. Boggio, * avec d'autres conditions aux limites, il est vrai.

Ces conditions correspondent au cas où l'on se donnerait, non les efforts exercés sur les bords, comme nous l'avons fait en général dans ce qui précède, mais les déplacements de ces bords. M. Boggio a, dans ces conditions, établi l'existence de la solution pour des catégories assez étendues de domaines. Il ne restait, on le voit, qu'à étendre ses résultats aux conditions aux limites (16).

VII.

Nous avons ainsi pu remplacer l'hypothèse consistant à admettre à priori qu'on est en présence d'une flexion par cette autre, irréprochable au point de

*Rendiconti dell'Academia delle Scienze di Torino, 17 décembre 1899; Nuovo Cimento, série 5, tome I, Mars 1901. Je saisis cette occasion pour indiquer que, dans une Note insérée aux Rendiconti dell' Instituto Lombardo (séance du 27 Juin 1901), M. Boggio a traité par des intégrales définies, le problème de l'équilibre de la plaque circulaire libre ou simplement appuyée, que j'ai étudié au même point de vue dans les Annales Scientifiques de l'Ecole Normale supérieure. Mes résultats avaient été exposée à la Société Mathématique de France dans la séance du 17 avril 1901, mais n'ont été publiés que dans le n° de Septembre de la même année.

vue des principes adoptés ici, que les efforts horizontaux ne sont pas très grands par rapport aux efforts verticaux.

Par contre, nous conservons encore dans notre raisonnement deux hypothèses que nous ne sommes pas en état de justifier.

C'est d'abord l'hypothèse de différentiation. Comme nous l'avons vu, celle-ci ne serait pas vérifiée si les diverses quantités que nous considérons présentaient des oscillations à très courtes périodes. Elle introduit donc la notion de ce qu'on pourrait appeler le *degré d'oscillation* de la quantité en question. C'est une notion évidemment analogue à celle qui intervient souvent, comme l'a fait remarquer M. BOREL,* en théorie des fonctions. L'une est dans le domaine de l'extrêmement petit ce que l'autre est dans le domaine de l'extrêmement grand. Il n'est pas douteux que la notion dont nous venons de parler ne soit fréquemment utile en Physique mathématique et surtout dans les questions d'approximation telles que celle qui nous occupe. La seule difficulté serait de choisir entre les diverses expressions (maximum des dérivées, intégrales portant sur les carrés de ces dérivées, variation au sens de M. JORDAN, etc.), susceptibles de mesurer ce degré d'oscillation.

Dans la théorie des fonctions harmoniques, on peut aisément évaluer les degrés d'oscillation. On sait, en effet, que si les valeurs d'une fonction harmonique sont limitées sur la surface d'une sphère, on peut assigner des limites supérieures pour les dérivées de cette fonction en tout point suffisamment éloigné de la surface. On pourrait chercher à étendre ce résultat aux équations de l'élasticité; mais, dans le problème que nous traitons en ce moment, une difficulté particulière naîtrait évidemment du fait que le domaine envisagé a une dimension extrêmement petite.

On est donc, en somme, ramené à une étude qui donnerait évidemment la solution de tous les problèmes relatifs aux plaques, à savoir celle de l'influence exercée par la forme d'un domaine sur les solutions des équations aux dérivées partielles à caractéristiques imaginaires considérées dans ce domaine. Cette étude reste encore à peu près entièrement à faire. Par exemple, les méthodes qui nous démontrent l'existence des solutions du problème de DIRICHLET ne nous renseignent guère sur la manière dont les résultats varient avec la forme du volume donné.

Nous retrouvons d'ailleurs cette même question à un autre point de vue. En disant en effet que la dimension verticale de notre plaque est extrêmement petite, nous entendons évidemment qu'elle est très petite par rapport aux dimensions horizontales. Il faudrait donc faire intervenir celles-ci dans les résultats, autrement dit, savoir comment elles influent sur la limitation des quantités u_0, v_0, solutions des équations. C'est un point sur lequel on est fixé dans une certaine

* *Mémoire sur les séries divergentes*, Annales Scientifiques de l'Ecole Normale Supérieure, 3e série, t. 16 (1899), ch. 1.

mesure, mais non d'une manière suffisante, par l'étude de ce qui se passe lorsqu'on remplace le contour donné de la plaque par un contour homothétique.

Remarquons enfin que notre hypothèse de différentiation ne peut manifestement subsister sans restriction, que si elle est vraie pour les efforts donnés, c'est à dire si les derivées de ceux-ci sont du même ordre qu'eux. Malheureusement, ceci cesse d'avoir lieu dans des cas très simples. Il est clair, par exemple, que si l'on pose un corps pesant sur la plaque, l'effort t_{zz} varie très rapidement sur les bords de la région ainsi pressée. Il serait donc nécessaire de modifier les raisonnements pour les étendre aux cas où de pareilles singularités se produiraient.

La seconde hypothèse que nous avons laissé subsister est celle d'après laquelle les efforts dans les parties de la plaque voisine des bords sont du même ordre que dans l'intérieur ou du moins d'après laquelle les résultantes de ces efforts suivant des sections telles que *abcd* (Fig. 1) ont l'ordre de grandeur indiqué par les équations (11).

Il n'est pas exact que, dans un corps en équilibre élastique les composantes de l'effort dans le voisinage de la surface soient nécessairement dans un rapport fini avec les valeurs de ces mêmes composantes a l'intérieur. Le contraire aurait lieu, d'après le principe de SAINT VENANT, si l'on faisait agir sur une petite région de la surface des forces très grandes se faisant équilibre. Si donc on n'écartait pas ici comme exceptionnel l'existence de telles forces, les conditions aux limites telles que nous les avons obtenues pourraient se trouver modifiées en certains points de contour. Mais il serait sans doute possible de démontrer (une fois établi le principe de SAINT VENANT) que ces modifications seraient sans influence sur les valeurs des inconnues dans l'intérieur de la plaque.

D'autre part nous avons fait implicitement une troisième hypothèse qui, toute classique qu'elle soit, il n'en est pas moins nécessaire de rappeler, en réduisant les tensions à leurs termes linéaires par rapport aux composantes de déformation. Rien n'est plus propre que le problème des plaques à montrer combien cette hypothèse mérite d'etre examinée. On sait, en effet, que si la plaque devient une membrane, ou simplement si elle est fortement tendue, les termes du second ordre s'introduisent nécessairement. Au reste, l'influence nécessaire de ces termes du second ordre sur les plaques est mise en évidence par ce fait d'expérience courante que les efforts horizontaux agissent d'une manière toute différente suivant qu'ils sont dilatants ou comprimants. *

* Certaines hypothèses sur les données du problème—par conséquent, légitimes au point de vue purement mathématique—mériteraient elles-mêmes d'être reprises. La première de toutes, celle qui consiste à regarder l'épaisseur de la plaque comme très petite relativement aux autres longueurs qui figurent dans la question est du nombre, car des feuilles extrêmement minces devraient être traitées comme des membranes et non comme des plaques, de sorte que des termes du second ordre y deviendraient prépondérants. L'expérience semble montrer qu'il existe des plaques assez épaisses pour que les termes du second ordre par rapport aux déformations puissent

On voit combien de lacunes graves présente encore la théorie précédente, surtout eu égard à la simplicité apparente de la question. Cependant, le lecteur qui prendra la peine de la comparer aux théories antérieurement proposées,—par exemple, à celle de SAINT VENANT, que j'ai suivie dans tout ce qui précède, parce qu'elle est l'étude la plus approfondie qui ait été écrite sur ce sujet—reconnaîtra sans doute qu'elle constitue un progrès notable sur ces dernières.

VIII.

Lorsque l'on considère une déformation *finie* de la plaque, l'hypothèse que cette déformation est une flexion doit être forcément adoptée, dans l'état actuel de la question. Si l'on l'abandonnait, en effet, on ne pourrait plus regarder les composantes de déformation comme très petites, et il faudrait introduire la théorie de l'équilibre élastique avec déformation finie.*

L'hypothèse en question doit donc être admise, et c'est d'elle que les résultats demandés découlent. En particulier, elle montre immédiatement que l'élément linéaire du feuillet moyen déformé est approximativement celui d'un plan.

On en conclut que ce feuillet moyen affecte approximativement la forme d'une surface développable.

Mais il est clair d'après ce qui précède que cette déduction soulève une question d'analyse qu'il serait intéressant de résoudre.

Elle ne peut, en effet, s'opérer que par des différentiations. Du moins, toutes les méthodes connues par lesquelles on démontre qu'une surface applicable sur un plan est nécessairement développable conduisent a différentier les coefficients de l'élément linéaire.

Si ces coefficients ne sont qu'approximativement égaux à l'unité ou à zéro, et si cette approximation ne se conserve pas dans la dérivation, peut on compter néanmoins que les méthodes dont nous venons de parler fournissent un résultat approché? Il est permis d'en douter, et de se demander si, dans ces conditions, les résultats obtenus ne seront pas analogues à ceux qui se présentent lorsque la différentiation des égalités *exactes* n'est pas permise et qui, comme l'a montré M. LEBESGUE, sont profondément différents des résultats classiques.

Toutefois, il est à remarquer qu'on peut arriver au but sans faire intervenir l'hypothèse de différentiation dans un cas, celui où il s'agit d'une déformation

être négligés, et cependant assez minces pour que les autres approximations faites dans ce qui précède soient légitimes ; mais, jusqu'ici, la théorie ne nous renseigne pas à cet égard.

D'autre part, nous avons supposé que les coefficients d'élasticité étaient tous du même ordre de grandeur. Dans le cas contraire, comme les rapports de certaines de ces coefficients viendraient dans quelques unes des calculs précédents, multiplier l'épaisseur de la plaque, il y aurait lieu de voir si quelques uns des résultats ne se trouveraient pas modifiés.

* Certaines parties des raisonnements précédents subsisteraient, même dans ce cas. Ainsi nous pourrions encore démontrer que les tensions verticales sont très petites, par rapport aux horizontales, et même en déduire l'expression de g_{xz}, g_{yz}, ϑ_z en fonction de ϑ_x, ϑ_y, g_{xy}.

infiniment petite, et où les composantes de déformation peuvent être réduites à leurs parties linéaires, soient, pour trois d'entre elles:

$$\frac{\partial u}{\partial x}, \qquad \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}.$$

Prenons, en effet, les équations

$$\frac{\partial u}{\partial x}=\theta\epsilon, \qquad \frac{\partial v}{\partial y}=\theta\epsilon, \tag{17}$$

$(|\theta|<1)$,

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\theta\epsilon \tag{18}$$

où ϵ désigne indifféremment plusieurs quantités du même ordre de grandeur.

Les deux premières donnent évidemment

$$u=f(y)+\theta\epsilon A, \qquad v=\phi(x)+\theta\epsilon A \qquad (|\theta|<1), \tag{19}$$

A étant la plus grande dimension de la plaque.

Il nous est impossible de différentier les deux équations précédentes pour obtenir les valeurs de $\partial u/\partial y$, $\partial v/\partial x$ afin de les transporter dans la relation (18). Nous intégrerons donc celle-ci daus l'aire d'un rectangle ayant ses côtés parallèles aux axes de coordonnées et représentés par les équations

$$x=x_1, \qquad x=x_2, \qquad y=y_1, \qquad y=y_2.$$

Il viendra

$$\int(udx-vdy)=\theta\epsilon(x_2-x_1)(y_2-y_1),$$

ou, en tenant compte de (19),

$$\begin{aligned}(x_2-x_1)[f(y_1)-f(y_2)]-(y_2-y_1)[\phi(x_2)-\phi(x_1)]\\ =\theta\epsilon(x_2-x_1)(y_2-y_1)+\theta\epsilon A(|x_2-x_1|+|y_2-y_1|).\end{aligned} \tag{20}$$

Par un point a du contour de la plaque, menons des parallèles ab, ac aux axes, jusqu'à nouvelle rencontre en b, c avec le contour (Fig. 2). La parallèle menée par le point b à l'axe des y coupera à nouveau le contour en d et la parallèle menée par c à l'axe des x, en d'. Nous allons, pour simplifier, supposer

1° que la plaque est convexe;

2° qu'elle est entièrement comprise à l'intérieur de la figure formée par les rectangles $abdc$, $acd'e'$ (Fig. 2) et les demi cercles décrits extérieurement à ces rectangles, sur ab, ac, cd, $d'b$ comme diamètres respectifs.

Soient x_0, x_0+B, x_0+B' les abscisses des points a, d', d; y_0, y_0+C', y_0+C, les ordonnées de ces mêmes points, et posons

$$\frac{\phi(x_0+B)-\phi(x_0)}{B}=r.$$

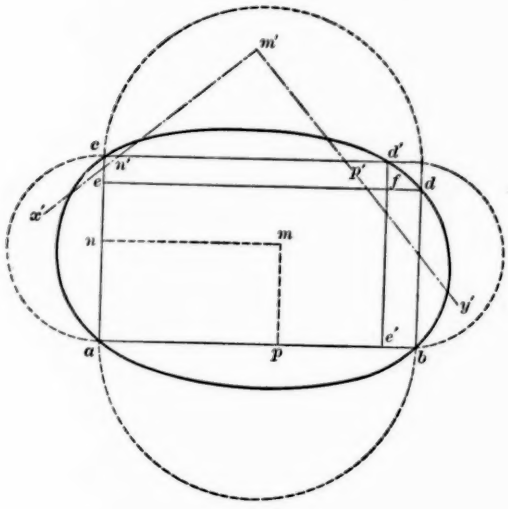

FIG. 2.

La relation (20), appliquée aux rectangles $abde$, $acd'e'$, $aee'f$ nous donnera

$$\frac{f(y_0 + C') - f(y_0)}{C'} = -r + \theta\epsilon + \theta\epsilon\left(\frac{A}{B} + \frac{A}{C'}\right),$$

$$\frac{f(y_0 + C') - f(y_0)}{C} = -r + \theta\epsilon\left(\frac{A}{B} + \frac{A}{C} + 1\right),$$

$$\frac{\phi(x_0 + B') - \phi(x_0)}{B'} = r + \theta\epsilon\left(\frac{A}{B} + 2\frac{A}{C} + \frac{A}{B'} + 2\right).$$

Si, de plus, m est un point quelconque intérieur à l'un des deux rectangles, il sera à une distance supérieure à $B/2$ de l'un au moins des côtés ac, bd, $d'e'$, et à une distance supérieure à $C/2$ de l'un au moins des côtés ab, de, $c'd'$.

La relation (20) appliquée à un rectangle tel que $amnp$ (Fig. 2) nous donnera dès lors

$$f(y) = f(y_0) - r(y - y_0),$$

$$\phi(x) = \phi(x_0) + r(x - x_0),$$

avec des erreurs qui seront dans un rapport fini avec ϵ.

Considérons enfin un point m' extérieur à nos rectangles: compris, par exemple, dans le cercle décrit sur cd comme diamètre. Par ce point, nous pourrons mener deux droites perpendiculaires entre elles et qui coupent toutes deux les rectangles. Soient x', y' des coordonnées rapportées à ces droites prises comme

axes; u', v' les projections des déplacements correspondants; α, l'angle de l'axe des x' avec l'axe des x. Les relations (17) donneront aisément

$$\left|\frac{\partial u'}{\partial x'}\right| < \epsilon, \qquad \left|\frac{\partial v'}{\partial y'}\right| < \epsilon. \tag{21}$$

Mais nous supposons que l'axe des x' contient un point n' intérieur à nos rectangles,—pour lequel, on a par conséquent

$$u = f(y_0) - r(y - y_0) - \epsilon D, \qquad v = \phi(x_0) + r(x - x_0) + \epsilon D$$

(D étant fini) ce qui permettra d'obtenir des valeurs approchées pour la quantité u';—et l'axe des y', un point p' pour lequel on pourra de même écrire une expression de v'.

Des valeurs ainsi obtenues pour u' au point n' et pour v' au point p', les relations (21) permettent évidemment de déduire les valeurs approchées pour u' et v' (et, par conséquent, pour u et v) en m'.

Le résultat est évidemment que u et v sont, dans toute l'étendue de la plaque, de la forme

$$u = f(y_0) - r(y - y_0) + \epsilon D,$$

$$v = \phi(x_0) + r(x - x_0) + \epsilon D,$$

D étant fini.

C'est bien la forme que nous devions trouver pour les déplacements u, v.

IX

Nous terminerons par deux remarques d'ordre purement analytique.

La solution des difficultés présentées par la formation des conditions aux limites réside, nous l'avons rappelé, dans ce fait qu'il se produit des déformations notables au voisinage des bords et négligeables partout ailleurs: ces déformations complémentaires satisfont à l'équation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = k^2 u, \tag{22}$$

où k est très grand (de l'ordre de l'inverse de l'épaisseur).

Lorsque le domaine considéré est circulaire, on reconnaît aisément que les solutions de cette équation diminuent très rapidement dès qu'on s'éloigne du contour.

On peut arriver à la même conclusion, dans le cas général, de la manière suivante.

Considérons une solution u de l'équation (22), les valeurs absolues de u sur le contour étant toutes inférieures à M. Soit O un point, dont la distance minimum au contour soit δ. L'équation (22) admettra la solution $J(ikr)/J(ik\delta)$,

où r est la distance d'un point quelconque (x, y) au point O et J la fonction de Bessel.

Or, sur tout le contour, r étant supérieur à δ et $|u|$ inférieur à M, les fonctions $u + MJ(ikr)/J(ik\delta)$, $u - MJ(ikr)/J(ik\delta)$ seront, l'une positive, l'autre négative. Il en sera donc de même à l'intérieur, et au point O, en particulier, l'on aura

$$|u| < \frac{M}{J(ik\delta)},$$

inégalité qui met bien en évidence la diminution rapide de $|u|$ dès que δ n'est plus très petit, si k est très grand.

X.

Le mode de croissance de la fonction $J(ix)$, pour x positif et très grand est celui d'une exponentielle.

Plus généralement, on peut, ainsi qu'il est bien connu, trouver un développement qui représente asymptotiquement $J(ix)$ pour les grandes valeurs de x.

Mais ce développement n'est que formel. Prolongé indéfiniment, il est divergent pour toute valeur de la variable.

M. Borel a perfectionné, dans ces dernières années, l'étude des developpements de cette espèce. On pourrait chercher a déduire de ses travaux une expression de $J(ix)$ mettant en évidence son mode de croissance, qui soit cependant convergente.

Les considérations suivantes (tout analogues à une démonstration bien connue de Weierstrass) conduiraient peut-être au même résultat.

Soit la série de puissances

$$a_0 + a_1 x + \cdots + a_m x^m + \cdots, \tag{23}$$

divergente pour toute valeur de x et que nous considérons pour x réel et positif (ou du moins en supposant que l'argument de x est compris entre deux limites suffisament voisines de zéro).

On a

$$e^{\frac{1}{x}} = 1 + \frac{1}{x} + \frac{1}{2!}\frac{1}{x^2} + \cdots + \frac{1}{m!}\frac{1}{x^m} + \cdots.$$

Soit $P_p(1/x)$ l'ensemble des premières termes de ce développement. La différence $1 - e^{-1/x} P_p(1/x)$ est plus petite en valeur absolue qu'un nombre ϵ, que l'on peut rendre aussi petit qu'on veut en prenant p suffisament grand si $|x|$ est supérieur à un nombre donné α.

Alors, en designant par α_i des nombres tendant vers zéro, par $a_i \epsilon_i$ les termes d'une série absolument convergente, et par p_i des entiers tels

$$\left|1 - e^{-\frac{1}{x}} P_{p_i}\left(\frac{1}{x}\right)\right|$$

soit inférieur à ϵ_i pour $|x| > \alpha_i$, la série

$$a_0 + a_1 x\left[1 - e^{-\frac{1}{x}} P_{p_1}\left(\frac{1}{x}\right)\right] + \cdots + a_m x^m\left[1 - e^{-\frac{1}{x}} P_{p_m}\left(\frac{1}{x}\right)\right] + \cdots$$

sera convergente et aura les mêmes valeurs asymptotiques que (23).

Cette méthode peut évidemment être variée d'une infinite de manières, en changeant les valeurs des entiers p_i et en remplaçant $e^{-1/x}$ par d'autres exponentielles. En changeant x en $1/x$ et appliquant à la série divergente qui intervient dans le calcul de $J(ix)$, on pourrait espérer obtenir une expression exacte de cette fonction.

COVARIANTS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS AND APPLICATIONS TO THE THEORY OF RULED SURFACES*

BY

E. J. WILCZYNSKI

In the following paper, the author wishes to present the theory of the covariants of a system of differential equations

$$(1)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

with the independent variable x. Some of these covariants present themselves, in a very elegant and simple form, in connection with the theory of the system adjoined to (1). The theory of covariants is here carried out to the same degree of completeness as has been previously done for the invariants. Then follows their geometrical interpretation.†

§ 1. *Fundamental properties of covariants.*

Any function of $y, z, y', z', \cdots, p_{ik}, p'_{ik}, \cdots, q_{ik}, q'_{ik}, \cdots$ which has the same value for system (1) and for any system equivalent to (1) by a transformation of the infinite group G,

$$(2)\quad \xi = \xi(x), \quad \eta = \alpha(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z, \quad \alpha\delta - \beta\gamma \neq 0,$$

shall be called an *absolute covariant*. In particular if it does not contain y, z, y', z', etc., it is called an *invariant*. The invariants of system (1) have already been determined.

Let us assign weights zero to y and z, unity to p_{ik} and two to q_{ik}. Moreover let every differentiation increase the weight by one, and let the weight of a

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† To facilitate references to my previous papers on this subject, I shall quote them in the text by the initial words of their titles, viz.: Invariants, Geometry, Reciprocal Systems. They have all been published in these Transactions: *Invariants*, vol. 2, p. 1; *Geometry*, vol. 2, p. 343; *Reciprocal Systems*, vol. 3, p. 60.

product be the sum of the weights of its factors. These conventions are somewhat different from those adopted in my paper on invariants, where I have used negative weights, following the example set by FORSYTH in a similar theory.* However the assignment of weights here made, which is essentially the same as that used by BOUTON,† appears after all to be more convenient.

In the proofs of the following theorems, I have utilized for my present purpose some ideas from the algebraic theory of quantics, as well as from the theory of invariants and covariants of a single linear differential equation.

We have already defined the term: absolute covariant. The function

$$C(y, z, y', z', \cdots, p_{ik}, p'_{ik}, \cdots, q_{ik}, q'_{ik}, \cdots)$$

is a *relative covariant*, if the equation

$$C = 0,$$

has as its consequence

$$\Gamma = 0,$$

where Γ denotes the same function of the transformed quantities

$$\eta, \zeta, \cdots, \pi_{ik}, \cdots, \rho_{ik}, \cdots,$$

as C of the corresponding original quantities.

Let us make the transformation

$$\xi = x, \qquad \eta = Cy, \qquad \zeta = Cz,$$

where C is a constant, and which is contained in the group G. Always denoting the transformed quantities by Greek letters, we shall then have

$$\eta^{(\lambda)} = Cy^{(\lambda)}, \qquad \zeta^{(\lambda)} = Cz^{(\lambda)}, \qquad \pi_{ik} = p_{ik}, \qquad \rho_{ik} = q_{ik}.$$

Therefore, any relative covariant must be homogeneous in y, z, y', z', etc. Any absolute covariant must be homogeneous in these quantities of degree zero.

Again, denoting by C a constant, let us make the transformation

$$\xi = Cx, \qquad \eta = y, \qquad \zeta = z,$$

which is also included in the group G. This gives

$$\eta^{(\lambda)} = C^{-\lambda}y^{(\lambda)}, \qquad \zeta^{(\lambda)} = C^{-\lambda}z^{(\lambda)}, \qquad \pi_{ik} = C^{-1}p_{ik}, \qquad \rho_{ik} = C^{-2}q_{ik},$$

which shows that every term of weight w is multiplied by C^{-w}.

A covariant must then be an isobaric function of the quantities upon which it depends, and of weight zero if it is an absolute covariant.

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*A. R. FORSYTH, Philosophical Transactions, vol. 179.

†C. L. BOUTON, American Journal of Mathematics, vol. 21 (1899).

Let $C_{\lambda, w}$ be an integral rational function of y, z, p_{ik}, q_{ik} and of the derivatives of these quantities, which is a covariant. Then it is necessarily homogeneous, say of degree λ in y, z, y', z', etc., and isobaric, say of weight w. Moreover, let $C_{\lambda, w}$ be irreducible in the sense that it is not possible to factor it into rational factors of lower weight and degree. Let us consider the effect upon $C_{\lambda, w}$ of a transformation of the dependent variables y and z alone.

Let the transformation be given in the form

$$y = \alpha_{11}\eta + \alpha_{12}\zeta, \qquad z = \alpha_{21}\eta + \alpha_{22}\zeta, \tag{3}$$

or

$$\Delta\eta = \alpha_{22}y - \alpha_{12}z, \qquad \Delta\zeta = -\alpha_{21}y + \alpha_{11}z, \tag{4}$$

$$\Delta = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21},$$

where α_{ik} are arbitrary functions of x.

If the coefficients of the transformed system of differential equations are again denoted by π_{ik} and ρ_{ik}, we shall have (*Invariants*, p. 3, if there we put $n = m = 2$),

$$\Delta\pi_{11} = 2(\alpha'_{11}\alpha_{22} - \alpha'_{21}\alpha_{12}) + p_{11}\alpha_{11}\alpha_{22} + p_{12}\alpha_{21}\alpha_{22} - p_{21}\alpha_{11}\alpha_{12} - p_{22}\alpha_{21}\alpha_{22},$$

. . . . . .

$$\Delta\rho_{11} = \alpha''_{11}\alpha_{22} - \alpha''_{21}\alpha_{12} + p_{11}\alpha'_{11}\alpha_{22} + p_{12}\alpha'_{21}\alpha_{22} - p_{21}\alpha'_{11}\alpha_{12} - p_{22}\alpha'_{21}\alpha_{22} + q_{11}\alpha_{11}\alpha_{22} + q_{12}\alpha_{21}\alpha_{22} - p_{21}\alpha_{11}\alpha_{12} - p_{22}\alpha_{21}\alpha_{22}, \tag{5}$$

. . . . . .

where only one equation of each kind has been written down.

Let $\Gamma_{\lambda, w}$ denote what $C_{\lambda, w}$ becomes when y, z, p_{ik}, etc., are replaced by η, ζ, π_{ik}, etc. Since $C_{\lambda, w}$ is a relative covariant, the equation $\Gamma_{\lambda, w} = 0$, must be a consequence of $C_{\lambda, w} = 0$. But equations (4), (5) and those deducible from these by differentiation, show clearly that in place of every term of weight w in C, we shall have in Γ a collection of terms of weight w plus terms of lower weight. But these latter terms must annihilate each other if $C_{\lambda, w}$ is a covariant, i. e., their sum must be identically zero. For they cannot vanish in consequence of $C_{\lambda, w} = 0$, since their aggregate is rational and of lower weight than w, while $C_{\lambda, w}$ cannot be factored into rational factors of lower weight. Now it is clear from equations (4) and (5) that the terms of weight w in $\Gamma_{\lambda, w}$ when expressed in terms of y, z, p_{ik}, etc., will contain only the quantities α_{ik} themselves and not their derivatives. We see then that there must be an equation of the form

$$\Gamma_{\lambda, w} = f(\alpha_{ik})C_{\lambda, w}, \tag{6}$$

where $f(\alpha_{ik})$ depends only upon the four arguments indicated.

Moreover, from (4) and (5) we notice that π_{ik} and ρ_{ik} are homogeneous functions of degree zero, and that η and ζ are homogeneous functions of degree -1 of the quantities α_{ik}, so that $\Gamma_{\lambda, w}$, and therefore $f(\alpha_{ik})$, will be a homogeneous function of degree $-\lambda$ of its arguments. Further it is clear, from (4) and (5), that $f(\alpha_{ik})$ can be written in the form

$$f(\alpha_{ik}) = \frac{\phi(\alpha_{ik})}{\Delta^{\mu}}, \tag{7}$$

where $\phi(\alpha_{ik})$ is an integral rational function of its arguments of degree $-\lambda + 2\mu$, since $f(\alpha_{ik})$ is of degree $-\lambda$. Therefore

$$\Gamma_{\lambda, w} = \frac{\phi(\alpha_{ik})}{\Delta^{\mu}} C_{\lambda, w}. \tag{8}$$

Just in the same way if we replace in $C_{\lambda, w}$ the quantities y, z, p_{ik}, etc., by their values in terms of η, ζ, π_{ik}, etc., obtained by solving (4) and (5), we should find

$$C_{\lambda, w} = \frac{\phi(A_{ik})}{\Delta'^{\mu}} \Gamma_{\lambda, w}, \tag{9}$$

where A_{ik} denotes the minor of α_{ik} in Δ divided by Δ, and where

$$\Delta' = A_{11}A_{22} - A_{12}A_{21} = \frac{1}{\Delta}. \tag{10}$$

From (8) and (9) we have by multiplication

$$\phi(\alpha_{ik})\phi(A_{ik}) = 1,$$

or writing out the four arguments

$$\phi(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})\phi\left(\frac{\alpha_{22}}{\Delta}, -\frac{\alpha_{12}}{\Delta}, -\frac{\alpha_{21}}{\Delta}, \frac{\alpha_{11}}{\Delta}\right) = 1,$$

or, since ϕ was homogeneous of degree $2\mu - \lambda$,

$$\phi(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})\phi(\alpha_{22}, -\alpha_{12}, -\alpha_{21}, \alpha_{11}) = \Delta^{2\mu-\lambda}.$$

But Δ cannot be factored into two integral rational factors,* so that the factors on the left member of this equation must themselves be powers of Δ. Therefore $f(\alpha_{ik})$ is a power of Δ. Moreover since $f(\alpha_{ik})$ is of degree $-\lambda$ while Δ is of the second degree, we shall have

$$\Gamma_{\lambda, w} = \Delta^{-\frac{\lambda}{2}} C_{\lambda, w},$$

* E. B. ELLIOTT, *Algebra of Quantics*, § 14.

for it becomes clear that a numerical factor k different from unity is inadmissible, by applying the identical transformation.

For our proof it was convenient to take the transformation in the form (3), solved for y and z. If we write instead the transformation in the form

$$\eta = \alpha y + \beta z, \qquad \zeta = \gamma y + \delta z,$$

we know now that a rational covariant $C_{\lambda, w}$ of weight w and of degree λ is transformed in accordance with the equation

$$\Gamma_{\lambda\ w} = (\alpha\delta - \beta\gamma)^{\frac{\lambda}{2}}\ C_{\lambda, w}.$$

Moreover, as the right member must obviously be rational in α, β, γ, δ, we get this theorem:

There are no rational covariants of odd degree for a binary system of linear homogeneous differential equations.

It is obvious how this theorem will generalize for m-ary systems.

Let us now make a transformation of the independent variable

$$\xi = \xi(x).$$

We shall find, by merely consulting the formulæ [*Invariants*, equations (56)], that every term of weight w in $\Gamma_{\lambda, w}$ is equal to a corresponding term of $C_{\lambda, w}$ multiplied by

$$(\xi')^{-w},$$

plus terms of lower weight. But the aggregate of the latter terms must vanish identically, since it cannot vanish as a consequence of the equation $C_{\lambda, w} = 0$, which is isobaric of weight w. Therefore we shall have

$$\Gamma_{\lambda, w} = \frac{1}{(\xi')^{w}}\ C_{\lambda, w}.$$

Combining our results we have the following fundamental theorem:

If $C_{\lambda, w}$ is an integral rational covariant of degree λ and of weight w, it is transformed by the transformation

$$\xi = \xi(x), \qquad \eta = \alpha(x)y + \beta(x)z, \qquad \zeta = \gamma(x)y + \delta(x)z,$$

in accordance with the equation

$$\Gamma_{\lambda, w} = \frac{(\alpha\delta - \beta\gamma)^{\frac{\lambda}{2}}}{(\xi')^{w}}\ C_{\lambda, w}. \tag{11}$$

Moreover its degree λ is necessarily even.

The product $C^{m_1}_{2\lambda_1, w_1} C^{m_2}_{2\lambda_2, w_2}$ is an absolute rational covariant, if and only if two integers m_1 and m_2, positive or negative, can be found satisfying the conditions

$$m_1\lambda_1 + m_2\lambda_2 = 0,$$
$$m_1 w_1 + m_2 w_2 = 0,$$

i. e., since the possibility $m_1 = m_2 = 0$ must be excluded, *an absolute covariant can be formed from* $C_{2\lambda_1, w_1}$ *and* $C_{2\lambda_2, w_2}$, *if and only if*

$$\lambda_1 w_2 - \lambda_2 w_1 = 0.$$

It follows at once that if two covariants of the same degree can be combined into an absolute covariant, their weights must be equal, and conversely.

§ 2. *Construction of covariants and semi-covariants.*

If we denote the left members of (1) by Y and Z, it is clear from our definitions of the transformed coefficients, that corresponding to the transformations

$$x = f(\xi), \qquad y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta,$$

Y and Z are transformed in accordance with the equations

$$(12) \qquad Y = \frac{\alpha\overline{Y} + \beta\overline{Z}}{(\xi')^2}, \qquad Z = \frac{\gamma\overline{Y} + \delta\overline{Z}}{(\xi')^2},$$

where $\overline{Y}$ and $\overline{Z}$ denote the left members of the transformed system of differential equations.

Since moreover the solutions U, V of the system of differential equations adjoined to (1) are transformed cogrediently with y and z (*Reciprocal Systems*, § 2), the equations (12) will also apply to the left members of the adjoined system

$$(13) \quad \begin{aligned} &U'' + p_{11}U' + p_{12}V' + \{q_{11} + \tfrac{1}{4}(u_{11} - u_{22})\}U + (q_{12} + \tfrac{1}{2}u_{12})V = 0, \\ &V'' + p_{21}U' + p_{22}V' + (q_{21} + \tfrac{1}{2}u_{21})U + \{q_{22} + \tfrac{1}{4}(u_{22} - u_{11})\}V = 0. \end{aligned}$$

Moreover this will still be true if in (13), we write y and z in place of U and V on account of the cogrediency of these quantities.

We have then at once two covariants of degree and weight 2. They are

$$(14) \quad C_1 = \begin{vmatrix} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z, & y \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z, & z \end{vmatrix},$$

$$C_2 = \begin{vmatrix} y'' + p_{11}y' + p_{12}z' + \{q_{11} + \tfrac{1}{4}(u_{11} - u_{22})\}y + (q_{12} + \tfrac{1}{2}u_{12})z, & y \\ z'' + p_{21}y' + p_{22}z' + (q_{21} + \tfrac{1}{2}u_{21})y + \{q_{22} + \tfrac{1}{4}(u_{22} - u_{11})\}z, & z \end{vmatrix}.$$

Of course C_1 vanishes for solutions of (1), and C_2 for (13). If we write

$$C = 2(C_2 - C_1) = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz. \tag{15}$$

C is also a covariant of weight and degree 2. This may, of course, be tested analytically.

The covariant C vanishes identically, i. e., for arbitrary functions y and z of x, if and only if the integrating ruled surface of system (1) is of the second order. For the necessary and sufficient conditions for this are (*Geometry*, p. 360),

$$u_{11} - u_{22} = u_{12} = u_{21} = 0.$$

We shall call a function of $y, z, y', z', p_{ik}, q_{ik}$, etc., a *semi-covariant* if it possesses the covariant property for transformations of the dependent variables y and z alone, whether or not it also has the covariant property for transformations of x.

Now v_{ik} and w_{ik} are cogredient with u_{ik} for transformations of the dependent variables (*Invariants*, pp. 8, 9, 10). Therefore

$$\begin{aligned} E &= v_{12}z^2 - v_{21}y^2 + (v_{11} - v_{22})yz, \\ F &= w_{12}z^2 - w_{21}y^2 + (w_{11} - w_{22})yz, \end{aligned} \tag{16}$$

are semi-covariants of weight three and four respectively, and both of the second degree.

If then we make the transformation

$$y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta, \qquad \Delta = \alpha\delta - \beta\gamma,$$

and denote the transformed functions by dashes, we shall have

$$C = \Delta\bar{C}, \qquad E = \Delta\bar{E}, \qquad F = \Delta\bar{F},$$

so that the Wronskians

$$\begin{aligned} (C'E) &= C'E - CE', \qquad (E'F) = E'F - EF', \\ (F'C) &= F'C - FC', \end{aligned} \tag{17}$$

are also semi-covariants obviously of degree 4.

We have further

$$\frac{C'}{C} = \frac{\bar{C}'}{\bar{C}} + \frac{\Delta'}{\Delta},$$

and from the fundamental equations

$$p_{11} + p_{22} = \bar{p}_{11} + \bar{p}_{22} - 2\frac{\Delta'}{\Delta},$$

whence

$$2\frac{C'}{C}+p_{11}+p_{22}=2\frac{C'}{C}+\bar{p}_{11}+\bar{p}_{22},$$

so that

$$G=2C'+(p_{11}+p_{22})C \tag{18}$$

is a new semi-covariant, and obviously in the same way

$$H=2E'+(p_{11}+p_{22})E, \qquad M=2F'+(p_{11}+p_{22})F. \tag{19}$$

It is obviously sufficient to consider only those semi-covariants and covariants involving no higher derivatives of y and z than the first. For if a covariant contains higher derivatives of y and z, we can express them in terms of y, z, y', z' by means of equations (1) and others deduced therefrom by differentiation.

So as to proceed in an orderly manner, let us first determine all independent semi-covariants containing besides y, z, y', z', merely the quantities p_{ik}, p'_{ik} and q_{ik}. We have already found one such, namely C. We can find another by forming $G-E$. If we make use of the definitions (*Invariants*, equations (32)), of the quantities v_{ik}, we shall have

$$\begin{aligned} N=G-E=&\{2p_{22}u_{12}+p_{12}(u_{11}-u_{22})\}z^2-\{2p_{11}u_{21}-p_{21}(u_{11}-u_{22})\}y^2\\ &+\{2p_{21}u_{12}-2p_{12}u_{21}+(p_{11}+p_{22})(u_{11}-u_{22})\}yz\\ &+4u_{12}zz'-4u_{21}yy'+2(u_{11}-u_{22})(yz'+y'z), \end{aligned} \tag{20}$$

a semi-covariant of degree 2 and weight 3.

If we examine the system of partial differential equations, of which the seminvariants and semi-covariants involving only the above variables are the solutions, we shall find that it consists of 12 independent equations with 16 independent variables. It has therefore four independent solutions. There must then be four seminvariants and absolute semi-covariants depending upon these 16 variables. Of these we know the seminvariants I and J, and the two relative semi-covariants C and N. There must be one more, which could be found by integrating the system of partial differential equations just mentioned. It can also be found in another manner. Put

$$\begin{aligned} \rho&=2y'+p_{11}y+p_{12}z,\\ \sigma&=2z'+p_{21}y+p_{22}z. \end{aligned} \tag{21}$$

Then it may be verified that, for transformations of the dependent variables, ρ and σ are cogredient with y and z. Therefore

$$P=z\rho-y\sigma=2(y'z-yz')+p_{12}z^2-p_{21}y^2+(p_{11}-p_{22})yz \tag{22}$$

is a semi-covariant of degree 2 and of weight 1. Moreover it is clear that C, P and N are independent of each other and of I and J, so that all semi-covariants involving only $y, z, y', z', p_{ik}, p'_{ik}, q_{ik}$ have been found. N can be written more simply,

$$N = 2(u_{12}z\sigma - u_{21}y\rho) + (u_{11} - u_{22})(z\rho + y\sigma). \tag{23}$$

If we wish to find all seminvariants and semi-covariants involving p''_{ik} and q'_{ik} besides the former variables, we set up the system of partial differential equations satisfied by them. It contains 24 independent variables and 16 independent equations. Therefore, there must be 8 such seminvariants and semi-covariants. But we know them already. They are

$$I,\ I',\ J,\ J',\ K;\qquad \frac{P}{C},\ \frac{N}{C},\ \frac{E}{C}; \tag{24}$$

for these are independent. That this is so becomes clear from the following argument. If there exists a relation between them E/C must certainly occur in it. For, suppose it did not. The relation could not also be free of all three of the quantities I', J', K, as we have already seen that I, J, P/C and N/C are independent. If it contains one of them, say K, let us solve for K and find

$$K = f\left(I, J, I', J', \frac{P}{C}, \frac{N}{C}\right).$$

But this is manifestly impossible if either or both of the last two arguments actually appear. For K does not contain y, z, y', z' while P/C and N/C are independent functions of these variables. But this equation is impossible even if P/C and N/C do not appear, since I, I', J, J', K are independent seminvariants, as I have shown on a former occasion.

If there is a syzygy between the quantities (24) it must, therefore, contain E/C. Suppose that it be solved for E/C. Then we should have

$$\frac{E}{C} = f\left(I, I', J, J', K, \frac{P}{C}, \frac{N}{C}\right).$$

But again the last two arguments cannot appear. For P and N are independent functions of y' and z', which are not contained in E/C at all. But even if they are suppressed, such an equation is impossible, for it would make the semi-covariant E/C equal to a seminvariant containing no y or z. Thus we have shown that the quantities (24) are independent.

Again making use of a system of partial differential equations, we notice that there are $32 - 20 = 12$ independent seminvariants and absolute semi-covariants involving the additional variables $p_{ik}^{(3)}$ and q_{ik}''. They are

$$I, I', I'', J, J', J'', K, K', L, \frac{P}{C}, \frac{N}{C}, \frac{E}{C}, \tag{25}$$

for these, it is easily seen. are independent. F/C, which obviously belongs to this same class of semi-covariants must then be expressible in terms of the other twelve. Moreover the syzygy which F/C and these quantities (25) verify must be independent of P/C and N/C since F/C does not depend upon y' or z'. It may be put into the form

$$f(C, E, F;\ I, I', I'', J, J', J'', K, K', L) = 0,$$

where f is homogeneous in C, E and F. We need not at present investigate this syzygy any farther.

No new semi-covariants will appear if we continue our search, and as we have seen in a former paper, all new seminvariants which appear are derivatives of I, J, K and L. (*Invariants*, p. 10.)

Let us put

$$\frac{C}{P} = \gamma, \qquad \frac{E}{P} = \epsilon, \qquad \frac{N}{P} = \nu. \tag{26}$$

Then any absolute covariant must be a function of the seminvariants I, J, K, L, of their derivatives and of γ, ϵ and ν. If we consider only such covariants, containing no higher derivatives of p_{ik} and q_{ik} then $p_{ik}^{(3)}$ and $q_{ik}^{(2)}$ respectively, we shall find them as solutions of the system of partial differential equations

$$\begin{aligned} &Y_1 f + \gamma \frac{\partial f}{\partial \gamma} + 2\epsilon \frac{\partial f}{\partial \epsilon} + 2\nu \frac{\partial f}{\partial \nu} = 0, \\ &Y_2 f - 4\gamma \frac{\partial f}{\partial \epsilon} + 2\gamma \frac{\partial f}{\partial \nu} = 0, \\ &Y_3 f = Y_4 f = Y_5 f = 0, \end{aligned} \tag{27}$$

where $Y_1 f \cdots Y_5 f$ are the operators defined in a former paper, being the left members of the system of partial differential equations satisfied by the invariants. [*Invariants*, equations (77).]

The equations (27) are independent and contain twelve independent variables. Therefore, there must be $12 - 5 = 7$ independent absolute invariants and covariants, or eight independent relative invariants and covariants.

Of these we know five to be invariants, viz.: θ_4, θ_6, θ_{10}, θ_{15} and θ_{18}. *The three covariants are*

$$C_1 = P, \qquad C_2 = C, \qquad C_3 = E + 2N, \tag{28}$$

where, as in the case of invariants, the subscript denotes the weight. The three covariants are all of degree two. *All other covariants can be expressed in terms of* C_1, C_2, C_3 *and invariants.* As for the invariants there are an infinite number of independent ones, but they are all functions of the seminvariants I, J, K, L and of their derivatives.

As absolute covariants, we may take

$$\frac{C_2^4}{\theta_4 C_1^4} \quad \text{and} \quad \frac{C_3^2}{\theta_4 C_1^2}. \tag{29}$$

for it is clear that the covariants (28) alone cannot be combined to give both absolute covariants, since they are all of the same degree but of different weights.

§ 3. *Geometrical interpretation of* ρ, σ *and* P.

If (y_k) and (z_k), for $(k = 1, 2, 3, 4)$, are the members of a simultaneous fundamental system of solutions of (1), we interpret y_k and z_k as the homogeneous coördinates of two points P_y and P_z. As x changes, P_y and P_z describe two curves C_y and C_z, and the line joining them L_{yz} generates the integrating ruled surface S of (1), which remains invariant under all transformations of the group G. (*Geometry*, § 2.)

If in (21) we put $y = y_k$, $z = z_k$, we obtain

$$\begin{aligned} \rho_k &= 2y_k' + p_{11}y_k + p_{12}z_k, \\ \sigma_k &= 2z_k' + p_{21}y_k + p_{22}z_k, \end{aligned} \qquad (k = 1, 2, 3, 4), \tag{30}$$

which quantities we may again interpret as the homogeneous coördinates of two points P_ρ and P_σ. Clearly, P_ρ is a point of the plane tangent to the integrating ruled surface S at P_y, and P_σ is a point of the plane tangent to S at P_z.

If the points P_y and P_z be transformed into two other points and $P_{\bar{y}}$ $P_{\bar{z}}$ of the line L_{yz} by the equations

$$y = \alpha\bar{y} + \beta\bar{z}, \qquad z = \gamma\bar{y} + \delta\bar{z}, \tag{31}$$

then, as shown in § 2, P_ρ and P_σ will be transformed cogrediently into $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$, where

$$\rho = \alpha\bar{\rho} + \beta\bar{\sigma}, \qquad \sigma = \gamma\bar{\rho} + \delta\bar{\sigma}. \tag{32}$$

Therefore, if system (1) be transformed into an equivalent system by transformation (31), and if the corresponding quantities $\bar{\rho}$ and $\bar{\sigma}$ be formed for it, the points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ will lie on the line $L_{\rho\sigma}$ joining P_ρ and P_σ just as the points $P_{\bar{y}}$ and $P_{\bar{z}}$ lie on the line L_{yz} joining P_y and P_z.

Thus, we have, by equations (30) *a straight line* $L_{\rho\sigma}$ *corresponding to every generator* L_{yz} *of* S. *Moreover there is a one-to-one correspondence between the points of these two lines, which we now propose to investigate. Of course the lines* $L_{\rho\sigma}$ *generate a second ruled surface, which we will denote by* S'.

If $p_{12} = p_{21} = 0$, the curves C_y and C_z described on S by P_y and P_z are asymptotic lines. (*Reciprocal Systems*, p. 69.) Equations (30) show that then the points P_ρ, P_σ are points of the tangents to these curves at P_y and P_z respectively. But as we can always reduce a system of form (1) to a form (for instance the semi-canonical form), for which $p_{12} = p_{21} = 0$, we see that P_ρ and P_σ are points on the tangents to the asymptotic lines which pass through P_y and P_z respectively.

Now, through each point of the generator L_{yz}, or g, of the surface S passes an asymptotic line. The tangents of all these asymptotic lines along g are the generators of the second kind of an hyperboloid, three of whose generators of the first kind are g, g' and g'', where g' and g'' are two generators of the surface S infinitesimally close to g. Let us call this hyperboloid H, *the hyperboloid osculating* S *along* g.

We have seen that, according to (30), we have a point P' corresponding to every point P on g. Moreover P' is a point on the tangent to the asymptotic line passing through P, and further the locus of all points P', corresponding to all of the points P of g, is a straight line $L_{\rho\sigma}$.

This line $L_{\rho\sigma}$ *must then be a generator of the same kind as* g *on the hyperboloid* H, *osculating the surface* S *at* g.

Omitting the subscripts, and differentiating (30), we find

$$\rho' = 2y'' + p_{11}y' + p_{12}z' + p'_{11}y + p'_{12}z,$$
$$\sigma' = 2z'' + p_{21}y' + p_{22}z' + p'_{21}y + p'_{22}z,$$

whence, making use of (1),

$$\rho' = -p_{11}y' - p_{12}z' + (p'_{11} - 2q_{11})y + (p'_{12} - 2q_{12})z,$$
$$\sigma' = -p_{21}y' - p_{22}z' + (p'_{21} - 2q_{21})y + (p'_{22} - 2q_{22})z.$$

If the values of y' and z' in terms of y, z, ρ, σ be substituted from (30), the following equations will be obtained

$$R = 2\rho' + p_{11}\rho + p_{12}\sigma = u_{11}y + u_{12}z,$$
$$S = 2\sigma' + p_{21}\rho + p_{22}\sigma = u_{21}y + u_{22}z \tag{33}$$

where R and S are merely abbreviations for the left members, and where the quantities u_{ik} are the same as those in my paper on *Invariants*, equations (20).

Equations (33) show that if the planes tangent to the ruled surface S', of which $L_{\rho\sigma}$ is a generator, are constructed at P_ρ and P_σ, they will intersect the line L_{yz} in the points $(u_{11}y_k + u_{12}z_k)$ and $(u_{21}y_k + u_{22}z_k)$ respectively. Or, in other words, the lines joining ρ_k with $u_{11}y_k + u_{12}z_k$, and σ_k with $u_{21}y_k + u_{22}z_k$ are tangents of the ruled surface S' at P_ρ and P_σ respectively.

In particular then, if $u_{12} = u_{21} = 0$. *the lines which are tangent to the asymptotic curves of the surface* S *at* P_y *and* P_z *are also tangent to the ruled surface* S'. But we can find a simpler and more fundamental interpretation of the conditions $u_{12} = u_{21} = 0$.

Consider three consecutive generators g_{-1}, g_0, g_1 of the ruled surface S. The hyperboloid H_0 osculating S along g_0 is determined by these three lines. On H_0 we have a line $L_{\rho\sigma}$, or for short h_0, which is the generator of S' corresponding to the generator g_0 of S. Consider a fourth generator g_2 of S consecutive to g_1. The lines g_0, g_1, g_2 determine the hyperboloid H_1 osculating S along g_1. There is upon it a line h_1 which is the next generator of S'. Any tangent to S' along h_0 must intersect h_0 and h_1. If it is at the same time tangent to an asymptotic line of S at any point of g_0 it must intersect the lines g_{-1}, g_0, g_1 also. Such a line must then intersect the five lines $g_{-1}, g_0, g_1, h_0, h_1$. But since h_0 is on the hyperboloid determined by g_{-1}, g_0, g_1 we may suppress h_0, since any line intersecting the first three will also intersect h_0. Therefore we can say that such a line must intersect the four lines g_{-1}, g_0, g_1, h_1. But any line intersecting g_{-1}, g_0, g_1 is a generator of the second kind of the hyperboloid H_0, and any line intersecting g_0, g_1, h_1 is a generator of the second kind of the hyperboloid H_1, since g_0, g_1, h_1 are generators of the first kind on H_1.

Therefore any line, which is tangent to an asymptotic curve of S *at a point of* g, *and which is at the same time tangent to the surface* S' *at a point of the generator of that surface corresponding to* g, *is common to two consecutive osculating hyperboloids of the surface* S. *Or, in other words, such a line intersects four consecutive generators of the surface* S.

There are, in general, two such lines, since four lines in space have two real, imaginary or coincident straight line intersectors.

Now we shall see in the next paragraph, that there exists, in general, two and only two points on each generator, such that if their loci be taken as fundamental curves C_y and C_z, the corresponding system of differential equations, will satisfy the conditions $u_{12} = u_{21} = 0$. *They are, therefore, the points at which tangents can be drawn to the surface, which have four consecutive points in common with it.*

CAYLEY * has called such a point of a surface, at which a four-point tangent can be drawn, *a flecnode*, the tangent itself the *flecnode tangent*, and the locus of all of the flecnodes, the *flecnode curve*. It should be noticed that the flecnode tangents are not in general tangents to the flecnode curve. They are however tangent to the asymptotic curve, which passes through the flecnode.

The flecnode curve intersects every generator of the surface twice, and if its two branches be taken as fundamental curves, the conditions $u_{12} = u_{21} = 0$ *will be satisfied for the corresponding system of differential equations.*

The problem of effecting such a transformation of (1) as to make $u_{12} = u_{21} = 0$, i. e., the problem of determining the flecnode curve will be solved in the next paragraph.

We have seen that $L_{\rho\sigma}$ is a line on the hyperboloid H osculating S along L_{yz}. The question naturally arises whether this generator of H is essentially distinguished geometrically from every other. The question is to be answered in the negative. For, if in the covariant P, we replace ρ and σ by $\bar{\rho} = \rho + \lambda y$ and $\bar{\sigma} = \sigma + \lambda z$ respectively, where λ is an arbitrary seminvariant function of x, the covariant is not changed, and $\bar{\rho}$, $\bar{\sigma}$ are, for transformations of the dependent variables, cogredient with y and z as well as ρ and σ. Moreover a change of the independent variable x transforms ρ and σ into $\bar{\rho}$ and $\bar{\sigma}$. But the line joining the points $\bar{\rho}_k$ and $\bar{\sigma}_k$ is an arbitrary generator of H belonging to the same set as L_{yz}. Therefore, it is properly this set of generators on H, rather than any one of them, which is covariantly connected with the surface.

Still the line $L_{\rho\sigma}$ is distinguished among these generators in a pseudo-geometrical manner. Suppose $p_{ik} = 0$, so that the curves C_y and C_z are asymptotic lines, the system (1) having been reduced to the semi-canonical form. Then

$$\rho = 2y', \qquad \sigma = 2z'.$$

Consider the three consecutive generators g_{-1}, g_0, g_{+1} of S as belonging to the values of x

$$x_0 - \Delta x, \qquad x_0, \qquad x_0 + \Delta x,$$

respectively, where Δx is an infinitesimal. Construct the tangent to C_y at P_y. It meets the three generators g_{-1}, g_0, g_{+1} since C_y is an asymptotic curve. The coördinates of the three points of intersection, A, B and C, are

$$y_k - y_k' \Delta x, \qquad y_k, \qquad y_k + y_k' \Delta x,$$

so that the point P_ρ, whose coördinates are proportional to y_k', is the harmonic conjugate of B with respect to A and C. Similarly along C_z. We may therefore imagine that the line $L_{\rho\sigma}$ is selected as follows: We consider three

* CAYLEY, *On the Singularities of Surfaces*, Mathematical Papers, vol. II, p. 29.

generators of the surface S corresponding to three values of x forming an arithmetical progression of common difference d. Upon the hyperboloid determined by these three lines we construct the generator which is the harmonic conjugate of the middle line with respect to the other two. As the common difference d approaches the limit zero the hyperboloid approaches as a limit the osculating hyperboloid and the fourth generator approaches as a limit the line $L_{\rho\sigma}$.

Consider the totality of osculating hyperboloids. It contains ∞^2 straight lines which therefore form a congruence. If we choose from the lines of this congruence a single infinity, one on each hyperboloid, we get a ruled surface covariant with S. The surface S' generated by $L_{\rho\sigma}$ is such a one. But obviously there are an infinite number of such surfaces. In particular we may take such lines of the congruence, one on each hyperboloid, which intersect a given line, and thus deduce from the given surface another belonging to a special linear complex. To accomplish this, we need merely set up the system of differential equations for

$$y + \lambda\rho, \qquad z + \lambda\sigma,$$

and determine λ so that this system may have an integrating ruled surface belonging to a special linear complex. This, it seems, should be of importance for the integration of a system of form (1). In fact, if λ has been so determined, the adjunction of λ to the domain of rationality, would reduce the transformation group (generalized in the sense of PICARD and VESSIOT), of system (1) to a subgroup leaving invariant the linear complex.

If system (1) already has a ruled surface belonging to a linear complex, we can thus deduce from it another belonging to a linear congruence, and if the ruled surface of (1) belongs to such a congruence in the first place, such a transformation will lead to a ruled surface of the second order.

We can obtain another congruence associated with the surface, and make similar remarks about it, in the following manner. The solutions U_k, V_k of the system adjoined to (1) were defined as the coördinates of the planes tangent to the surface S at P_y and P_z respectively. (*Reciprocal Systems*, §§ 1 and 2.) The relations (*Reciprocal Systems*, equations (27))

$$\sum y_k U_k = 0, \qquad \sum z_k U_k = 0, \qquad \sum y'_k U_k = 0,$$

$$\sum y_k V_k = 0, \qquad \sum z_k V_k = 0, \qquad \sum z'_k V_k = 0,$$

show this. But we may also interpret U_k and V_k as point coördinates. Then, these equations show that U_k and V_k are the poles of the planes tangent to S at P_y and P_z with respect to the surface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

We have again a projective correspondence between the points of the line L_{yz} and its polar L_{UV}. If we join corresponding points of the two lines we obtain a hyperboloid. There is one such for every generator of S, and clearly we have again a congruence, made up of the first set of generators of these hyperboloids, to which the surface S belongs.

It is not my present purpose to follow out these ideas any further. I shall, however, deduce the system of differential equations of which ρ_k and σ_k are the solutions, and whose integrating ruled surface is S'.

If we solve (33) for y and z, we obtain

$$\begin{aligned} Jy &= u_{22}R - u_{12}S, \\ Jz &= -u_{21}R + u_{11}S, \end{aligned} \qquad J = u_{11}u_{22} - u_{12}u_{21}, \tag{34}$$

If (33) be differentiated, and the values of y' and z' in terms of y, z, ρ, σ be substituted the following equations result:

$$2R' = u_{11}\rho + u_{12}\sigma + (2u'_{11} - u_{11}p_{11} - u_{12}p_{21})y + (2u'_{12} - u_{11}p_{12} - u_{12}p_{22})z,$$
$$2S' = u_{21}\rho + u_{22}\sigma + (2u'_{21} - u_{21}p_{11} - u_{22}p_{21})y + (2u'_{22} - u_{21}p_{12} - u_{22}p_{22})z.$$

The quantities u'_{ik} may be expressed in terms of the quantities p_{ik} and v_{ik} (*Invariants*, equations (32)). If this be done, if, moreover, the equations be multiplied by J on both members, and if use be made of (34), the equations become

$$\begin{aligned} 2JR' - Ju_{11}\rho - Ju_{12}\sigma + t_{11}R + t_{12}S = 0, \\ 2JS' - Ju_{21}\rho - Ju_{22}\sigma + t_{21}R + t_{22}S = 0, \end{aligned} \tag{35}$$

where we have put

$$\begin{aligned} t_{11} &= Jp_{11} + u_{21}v_{12} - u_{22}v_{11}, \\ t_{12} &= Jp_{12} - u_{11}v_{12} + u_{12}v_{11}, \\ t_{21} &= Jp_{21} + u_{21}v_{22} - u_{22}v_{21}, \\ t_{22} &= Jp_{22} - u_{11}v_{22} + u_{12}v_{21}. \end{aligned} \tag{36}$$

Performing the differentiations indicated, inserting the values of R and S from (33), and collecting terms, we find the required system of differential equations:

$$\begin{aligned} &4J\rho'' + 2(Jp_{11} + t_{11})\rho' + 2(Jp_{12} + t_{12})\sigma' + (2Jp'_{11} - Ju_{11} \\ &\qquad + t_{11}p_{11} + t_{12}p_{21})\rho + (2Jp'_{12} - Ju_{12} + t_{11}p_{12} + t_{12}p_{22})\sigma = 0, \\ &4J\sigma'' + 2(Jp_{21} + t_{21})\rho' + 2(Jp_{22} + t_{22})\sigma' + (2Jp'_{21} - Ju_{21} \\ &\qquad + t_{21}p_{11} + t_{22}p_{21})\rho + (2Jp'_{22} - Ju_{22} + t_{21}p_{12} + t_{22}p_{22})\sigma = 0. \end{aligned} \tag{37}$$

Suppose that system (1) has the semi-canonical form, so that $p_{ik} = 0$. In general (37) will not then also assume the semi-canonical form. It will, if and only if $t_{ik} = 0$ together with $p_{ik} = 0$. But then, as equations (36) show, either $J = 0$, or $v_{11} = v_{12} = v_{21} = v_{22} = 0$. But if J were zero, equations (33) show that the planes tangent to S' at P_ρ and P_σ would intersect L_{yz} in the same point. But this can happen only in one of two ways. Either these planes are identical, in which case the surface S' is developable, or else the generator $L_{\rho\sigma}$ of the surface S' intersects L_{yz}. This latter case is only possible for singular values of x for which the osculating hyperboloid degenerates. For else the surface S would be developable, which it can never be. (*Geometry*, p. 347.) The case that S' is developable must also be excluded for the same reason. For then ρ and σ could verify no system of differential equations of the form here considered.

In order that (1) and (37) may be simultaneously put into the semi-canonical form, it is therefore necessary to have

$$p_{ik} = v_{ik} = 0, \tag{38}$$

whence u_{ik} and therefore q_{ik} are seen to be constants. Conversely, if $p_{ik} = 0$ and q_{ik} are constants, we find $v_{ik} = t_{ik} = 0$, so that (1) and (37) will both have the semi-canonical form, but from (38) follows also $w_{ik} = 0$, where the quantities w_{ik} have been defined in my paper on Invariants, equations (39). But these conditions make all the minors of the second order in the determinant

$$\Delta = \begin{vmatrix} u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{22}, & v_{12}, & v_{21} \\ w_{11} - w_{22}, & w_{12}, & w_{21} \end{vmatrix}$$

vanish, *so that the surface S, and consequently also S', is contained in a linear congruence.* (*Geometry*, p. 356.)

Let us notice that in the case here considered, $p_{ik} = v_{ik} = 0$, not only do (1) and (37) simultaneously assume the semi-canonical form, but they are absolutely identical, so that *the surfaces S and S' are projective transformations of each other.*

Not every ruled surface of a linear congruence has the property in question. We have found that the necessary and sufficient conditions for this property are, that if (1) be written in the semi-canonical form ($p_{ik} = 0$), the coefficients q_{ik} become constants. Now suppose that the directrices of the congruence are distinct. Then since they are asymptotic lines of the surface, we may choose them as fundamental curves C_y and C_z. Our system (1) then assumes the form

$$y'' + q_{11} y = 0, \qquad z'' + q_{22} z = 0,$$

if the directrices be taken as opposite edges of the tetrahedron of reference. Moreover q_{11} and q_{22} are constants. Assuming the edge C_y of the tetrahedron to be the intersection of the planes $x_3 = 0$ and $x_4 = 0$, and the edge C_z the intersection of $x_1 = 0$, $x_2 = 0$, we find by elementary integration, if q_{11} and q_{22} are different from zero,

$$y_1 = e^{r_1 x}, \qquad y_2 = e^{r_2 x}, \qquad y_3 = 0, \qquad y_4 = 0,$$

$$z_1 = 0, \qquad z_2 = 0, \qquad z_3 = e^{r_3 x}, \qquad z_4 = e^{r_4 x},$$

where

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \pm\sqrt{-q_{11}}, \qquad \left.\begin{matrix} r_3 \\ r_4 \end{matrix}\right\} = \pm\sqrt{-q_{22}}.$$

If we put $\eta_k = ay_k + bz_k$, where a and b are constants, $\eta_1 \cdots \eta_4$ are the coördinates of a point on an arbitrary asymptotic line of the surface. This gives

$$\eta_1 = ae^{r_1 x}, \qquad \eta_2 = ae^{r_2 x}, \qquad \eta_3 = be^{r_3 x}, \qquad \eta_4 = be^{r_4 x}.$$

The relation between $\eta_1 \cdots \eta_4$ obtained by eliminating a, b from these equations, will be the equation of the ruled surface S. It is

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}^{r_2 - r_4} \begin{pmatrix} \eta_3 \\ \eta_4 \end{pmatrix}^{-(r_1 - r_2)} = 1,$$

or

$$\eta_1^{2\sqrt{-q_{22}}}\eta_4^{2\sqrt{-q_{11}}} - \eta_2^{2\sqrt{-q_{22}}}\eta_3^{2\sqrt{-q_{11}}} = 0.$$

But all such surfaces admit at least a two-parameter group of projective transformations.*

We have assumed q_{11} and q_{22} different from zero. If $q_{11} = q_{22} = 0$, S' is a ruled surface of the second order, admitting a six-parameter projective group.

Suppose $q_{11} = 0$, $q_{22} \neq 0$. Then we have

$$y_1 = x, \qquad y_2 = 1, \qquad y_3 = 0, \qquad y_4 = 0,$$

$$z_1 = 0, \qquad z_2 = 0, \qquad z_3 = e^{r_3 x}, \qquad z_4 = e^{r_4 x}.$$

The equation of the surface becomes

$$\frac{\eta_3}{\eta_4} = e^{2\sqrt{-q_{22}}\frac{\eta_1}{\eta_2}},$$

and this admits a three parameter group of projective transformations.

* LIE, *Theorie der Transformationsgruppen*, vol. III, p. 198.

Suppose now that the directrices of the congruence coincide. Taking the directrix as C_y and an asymptotic line for C_z, we may assume the system to have the form

$$y'' = 0, \qquad z'' = ky,$$

where k is a constant. (*Geometry*, p. 358.) A fundamental system of solutions is given by

$$y_1 = x, \qquad y_2 = 1, \qquad y_3 = 0, \qquad y_4 = 0.$$
$$z_1 = \tfrac{1}{6}kx^3, \qquad z_2 = \tfrac{1}{2}kx^2, \qquad z_3 = x, \qquad z_4 = 1.$$

The equation of S is found to be

$$\eta_1\eta_4^2 - \eta_2\eta_3\eta_4 + \tfrac{1}{8}k\eta_3^3 = 0,$$

which by the transformation

$$\eta_1 = x_3, \qquad \eta_2 = 3\sqrt[3]{\frac{k}{6}}x_2, \qquad \eta_3 = \sqrt[3]{\frac{6}{k}}x_1, \qquad \eta_4 = x_4,$$

becomes

$$x_3x_4^2 - 3x_1x_2x_4 + 2x_1^3 = 0,$$

or CAYLEY'S cubic scroll, which admits a three-parameter group of projective transformations.

We have proved the following theorem. *If the curves on the surface S', corresponding to the asymptotic lines on S, are also asymptotic lines, then S must be contained in a linear congruence, and must admit at least a two-parameter group of projective transformations.*

We conclude the consideration of the covariant P by showing that it cannot vanish identically. For, suppose P were zero for all values of x. Then would

$$\rho_k = \lambda y_k, \qquad \sigma_k = \lambda z_k,$$

i. e., the surfaces S and S' would coincide generator for generator. Moreover, if we introduce the expressions for ρ_k and σ_k, we would have,

$$2y_k' + (p_{11} - \lambda)y_k = -p_{12}z_k,$$
$$2z_k' + (p_{22} - \lambda)z_k = -p_{21}y_k,$$

i. e., the generator of the surface would be tangent to both C_y and C_z at every pair of corresponding points, which is manifestly impossible unless the surface degenerates into a straight line.

The singular generators which are tangent to C_y and C_z simultaneously are the ones for which P can vanish.

§ 4. *The covariant C.*

Let us put

$$\eta = \tfrac{1}{2}uy + u_{12}z,$$

(39)

$$\zeta = u_{21}y - \tfrac{1}{2}uz,$$

where

$$u = u_{11} - u_{22}. \tag{40}$$

Then

$$C = \eta z - \zeta y. \tag{41}$$

If in (39) we put $y = y_k$, $z = z_k$, we have the coördinates of two new points on the line L_{yz}, dividing the segment $P_y P_z$ in the ratios $2u_{12} : u$ and $-u : 2u_{21}$ respectively. Denote them by P_η and P_ζ. These points are distinct if

$$\theta_4 \neq 0.$$

We may consider that (P_η, P_y) and (P_ζ, P_z) are two pairs of an involution on the line L_{yz}. Denote by λ and λ' the ratios in which any point of the line L_{yz}, and that corresponding to it in the involution divide the segment $P_y P_z$. Then the parametric equation of the involution will be

$$2u_{21}\lambda\lambda' + u(\lambda + \lambda') - 2u_{12} = 0. \tag{42}$$

The double points of the involution are determined by the quadratic equation obtained if in (42) λ' is put equal to λ. Therefore, they divide $P_y P_z$ in the ratios

$$-\frac{u + \sqrt{\theta_4}}{2u_{21}} = \frac{2u_{12}}{u - \sqrt{\theta_4}} \quad \text{and} \quad -\frac{u - \sqrt{\theta_4}}{2u_{21}} = \frac{2u_{12}}{u + \sqrt{\theta_4}}, \tag{43}$$

respectively. Of course, we may also find the double points by determining the pair of points dividing both $P_y P_\eta$ and $P_z P_\zeta$ harmonically.

If then we put

$$Y = \frac{u - \sqrt{\theta_4}}{2} y + u_{12}z = \rho\left[u_{21}y - \frac{u + \sqrt{\theta_4}}{2} z\right],$$

(44)

$$Z = \frac{u + \sqrt{\theta_4}}{2} y + u_{12}z = \sigma\left[u_{21}y - \frac{u - \sqrt{\theta_4}}{2} z\right],$$

where

$$\rho = \frac{u - \sqrt{\theta_4}}{2u_{21}} = -\frac{2u_{12}}{u + \sqrt{\theta_4}}, \qquad \sigma = \frac{u + \sqrt{\theta_4}}{2u_{21}} = -\frac{2u_{12}}{u - \sqrt{\theta_4}}, \tag{45}$$

Y_k and Z_k are the coördinates of the double points of the involution. In the case of real coefficients, it may therefore be classified as hyperbolic, parabolic or elliptic according as

$$\theta_4 \gtreqless 0.$$

But

$$YZ = u_{12}C, \tag{46}$$

so that the double elements of the involution may be determined by factoring the covariant C.

Let us examine this involution more in detail. If $u_{12} = u_{21} = 0$, the parametric equation of the involution (42), reduces to $\lambda + \lambda' = 0$ provided that $u \neq 0$. Therefore, if $u_{12} = u_{21} = 0$ the involution is referred to its double points as fixed elements. It is worth while to show directly that $u_{12} = u_{21} = 0$ if the involution is referred to its double points. If y and z are transformed by the equations

$$y = \alpha\bar{y} + \beta\bar{z}, \qquad z = \gamma\bar{y} + \delta\bar{z}, \qquad \alpha\delta - \beta\gamma = \Delta, \tag{47}$$

the quantities u_{ik} for the transformed system become $\bar{u}_{ik}$, where

$$\begin{aligned}
\Delta\bar{u}_{11} &= \alpha\delta u_{11} + \gamma\delta u_{12} - \alpha\beta u_{21} - \beta\gamma u_{22},\\
\Delta\bar{u}_{12} &= \beta\delta u_{11} + \delta^2 u_{12} - \beta^2 u_{21} - \beta\delta u_{22},\\
\Delta\bar{u}_{21} &= -\alpha\gamma u_{11} - \gamma^2 u_{12} + \alpha^2 u_{21} + \alpha\gamma u_{22},\\
\Delta\bar{u}_{22} &= -\beta\gamma u_{11} - \gamma\delta u_{12} + \alpha\beta u_{21} + \alpha\delta u_{22}.
\end{aligned} \tag{48}$$

This may be seen without much calculation by making use of the formulæ for the transformation of the quantities p'_{ik}, and also of the remark that the derivatives of α, β, γ, δ cannot enter into equations (48), as is shown by the formulæ for the infinitesimal transformations of the quantities u_{ik}, which have been deduced on a former occasion (*Invariants*, equations (37)).

Now the double points of the involution are given by

$$Y = \frac{u - \sqrt{\theta_4}}{2}y + u_{12}z, \qquad Z = u_{21}y - \frac{u - \sqrt{\theta_4}}{2}z,$$

whence

$$\Delta y = -\frac{u - \sqrt{\theta_4}}{2}Y - u_{12}Z, \qquad \Delta z = -u_{21}Y + \frac{u - \sqrt{\theta_4}}{2}Z,$$

where

$$\Delta = \tfrac{1}{2}\sqrt{\theta_4}(u - \sqrt{\theta_4}).$$

If $\Delta \neq 0$, we can put in (48),

$$\alpha = \frac{u - \sqrt{\theta_4}}{2}, \qquad \beta = u_{12}, \qquad \gamma = u_{21}, \qquad \delta = -\frac{u - \sqrt{\theta_4}}{2},$$

which gives $\bar{u}_{12} = \bar{u}_{21} = 0$, as we wished to prove.

If $\Delta = 0$, we have either $\theta_4 = 0$, i. e., the double points of the involution coincide, or else

$$\theta_4 = u^2,$$

whence $u_{12}u_{21} = 0$, and therefore either u_{12} or u_{21} vanish in the first place, say $u_{12} = 0$, and P_y is already one of the double points. But this can always be avoided except in one case. For u_{12} cannot vanish identically for all curves on the surface unless $\bar{u}_{12} = 0$ follows from $u_{12} = 0$ for all values of the arbitrary functions $\alpha, \beta, \gamma, \delta$. But this is so if, and only if, $u_{11} - u_{22}$ and u_{21} also vanish, i. e., if, and only if, the surface is of the second order.

Therefore, *if the surface is not of the second order, the double points of the involution determined by* (39) *on every generator of the surface, are the intersections of that generator with the flecnode curve. In the case of a system with real coefficients this curve is real for those generators for which* $\theta_4 > 0$, *and imaginary for those generators for which* $\theta_4 < 0$. *If* θ_4 *vanishes identically, the flecnode curve intersects every generator in two coincident points. If* θ_4 *does not vanish identically, the values of* x *for which* $\theta_4 = 0$ *correspond to generators which intersect the curve in two coincident points, i. e., to generators which are either tangent to the curve or upon which there is a double point of the curve.*

The involution becomes indeterminate, as equations (39) show, if and only if $u = u_{12} = u_{21} = 0$, i. e., if the surface is of the second order. That this must be so is obvious geometrically. But even if u, u_{12}, u_{21} do not vanish identically, there will be in general particular values of x for which they do. As a matter of fact, if θ_4 does not vanish identically, let us take as our fundamental curves C_y and C_z the two branches of the flecnode curve. Then $u_{12} = u_{21} = 0$, and $u_{11} - u_{22} = u$ will not vanish identically, since $\theta_4 = u^2$. But unless θ_4 is a constant, there will always be values of x for which $u = 0$, so that there will then be values of x for which $u_{12} = u_{21} = u_{11} - u_{22} = 0$ simultaneously. Along such generators the involution is indeterminate. The osculating hyperboloid has more than three consecutive generators in common with S. We shall say that it *hyperosculates* the surface. Therefore, *every ruled surface, for which* θ_4 *is not a constant, has a certain number of generators along which the osculating hyperboloid hyperosculates the surface. The parameters of such generators are found from the equations*

$$u_{11} - u_{22} = u_{12} = u_{21} = 0.$$

This singularity of ruled surfaces was mentioned to my knowledge for the first time by Voss,* who also was the first to pay much attention to the flecnode curve.

* Voss, *Zur Theorie der windschiefen Flächen*, Mathematische Annalen, vol. 8.

The flecnode curve may be an asymptotic line. But if it is, we have simultaneously

$$u_{12} = u_{21} = 0, \qquad p_{12} = p_{21} = 0,$$

whence follows also

$$q_{12} = q_{21} = 0,$$

so that in this case it consists of a pair of straight lines, which of course must not be coplanar.

Thus, if the flecnode curve is also an asymptotic line, it breaks up into a pair of distinct non-intersecting or coincident lines. The ruled surface then belongs to a linear congruence with distinct or coincident directrices, and these directrices are flecnode curves and asymptotic lines of the surface at the same time. If only one branch of the flecnode curve is an asymptotic line, then it is a straight line on the surface, and the surface belongs to a special linear complex.

The involution, which we have considered in this paragraph, has been considered before in the special case just mentioned, when the surface belongs to a linear congruence. According to a theorem first proved, I believe, by CREMONA,* we have an involution on every generator of such a surface determined as follows: *Every asymptotic line intersects every generator in two points. These points are the pairs of an involution whose double points are the intersections of the generator with the directrices of the congruence.*

This theorem enables us to give a concrete geometrical interpretation for the involution in the general case. Along a given generator g of the ruled surface S, construct the two flecnode tangents. Regard these as the directrices of a linear congruence. Consider any ruled surface, not of the second order, belonging to this congruence which has the generator g in common with S. Then every asymptotic line of this surface intersects g twice, and these two points of intersection constitute a pair of the involution.

We have seen how to transform the system of differential equation (1), so as to make $u_{12} = u_{21} = 0$. Equations (48) show that we may instead make $u_{11} - u_{22} = 0$. From (39) then follows at once, that the curves C_y and C_z are harmonic conjugates with respect to the two branches of the flecnode curve, if $u_{11} - u_{22} = 0$.

We have already noted that C vanishes identically, i. e., for absolutely arbitrary functions y and z, if and only if the integrating ruled surface is of the second order. But we can reduce the conditions under which this theorem holds. *If the covariant C vanishes for all possible curves C_y and C_z on the surface, the surface is of the second order.*

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*CREMONA, *Rappresentazione di una classe di superficie gobbe sopra un piano e determinazione delle loro curve assintotiche*, Annali di Mathematica, ser. 2, vol. 1 (1867–68).

For, suppose if possible that $C = 0$ for all curves C_y and C_z on S, and that $\theta_4 \neq 0$. Then the flecnode curve has two distinct branches, and, taking them as the fundamental curves, we shall have $u_{12} = u_{21} = 0$, whence since $C = 0$ follows $u_{11} - u_{22} = 0$, i. e., $\theta_4 = 0$. Therefore θ_4 cannot be different from zero. Let $\theta_4 = 0$ together with $C = 0$. We can certainly assume $u_{12} = 0$ by taking C_y to be the flecnode curve, unless the surface is of the second order, in which case $u_{12} = u_{21} = u_{11} - u_{22} = 0$ for all curves on the surface. But if $u_{12} = 0$ and $\theta_4 = 0$, we find at once $u_{11} - u_{22} = 0$, and since $C = 0$ also $u_{21} = 0$. Therefore C vanishes, as was stated, if and only if the surface S is of the second order.

Of course, there is an involution of planes through every generator of a ruled surface corresponding to the involution of points considered in this paragraph.

§ 5. *On the validity of certain transformations.*

In a former paper (*Geometry*, p. 347), it has been noted that the values of x corresponding to cuspidal generators are those, for which

$$D = \begin{vmatrix} y_1', & z_1', & y_1, & z_1 \\ y_2', & z_2', & y_2, & z_2 \\ y_3', & z_3', & y_3, & z_3 \\ y_4', & z_4', & y_4, & z_4 \end{vmatrix} = 0.$$

This however should be taken with a proviso. For, since $y_1 \cdots y_4$ have been interpreted as the homogeneous coördinates of a point, it is not admissible that they should vanish or become infinite simultaneously. For, in such cases, the point would be indeterminate. We must therefore assume that $y_1 \cdots y_4$, and similarly $z_1 \cdots z_4$ do not become zero or infinite simultaneously. This condition may always be fulfilled by multiplying y and z by properly chosen functions of x. Let us assume that this condition is fulfilled, and let $D = 0$ for $x = a$. Then there is a pinchpoint on the generator corresponding to $x = a$.

We can furnish a new proof of the fact that D must vanish for the values of x corresponding to cuspidal generators of the surface.

As in *Reciprocal Systems*, equations (5), let us write

$$D = \sum_{k=1}^{4} y_k' v_k = -\sum_{k=1}^{4} z_k' u_k.$$

Then u_k and v_k are the coördinates of the planes tangent to S at P_y and P_z respectively. If P_y and P_z are transformed into $P_{\bar{y}}$ and $P_{\bar{z}}$ by the equations

$$\bar{y} = \alpha y + \beta z, \qquad \bar{z} = \gamma y + \delta z,$$

these planes are transformed into the planes tangent to S at $P_{\bar{y}}$ and $P_{\bar{z}}$ by the equations

$$\bar{u} = \Delta(\alpha u + \beta v), \qquad \bar{v} = \Delta(\gamma u + \delta v),$$

where

$$\Delta = \alpha\delta - \beta\gamma.$$

Therefore if $v_k = \rho u_k$, i. e., if the tangent planes at P_y and P_z coincide, $\bar{u}_k$ and $\bar{v}_k$ are also proportional to u_k, *i. e., if the tangent planes at two points of a generator of the ruled surface coincide, it must be a cuspidal generator*, for the surface then has the same tangent plane at all points of the generator.

But then

$$D = \sum_{k=1}^{4} y_k' v_k = \rho \sum_{k=1}^{4} y_k' u_k = 0,$$

since u_k is the minor of z_k' in the determinant D.

We have shown (*Geometry*, equation (11)), that

$$\frac{1}{D}\frac{dD}{dx} = -(p_{11} + p_{22}), \tag{49}$$

whence, if $D \neq 0$,

$$D = Ce^{-\int (p_{11} + p_{22})\,dx}, \tag{50}$$

where C is a constant different from zero. Since by a transformation

$$y = \eta e^{-\frac{1}{2}\int p_{11}\,dx}, \qquad z = \zeta e^{-\frac{1}{2}\int p_{22}\,dx}, \tag{51}$$

we can always make p_{11} and p_{22} vanish, we can make D a constant, and thus apparently lose sight of the pinchpoints. But clearly such a transformation (51) is strictly legitimate only for regions of the surface which contain no cuspidal generators. For, suppose that $x = a$ corresponds to a cuspidal generator, and neither $(y_1 \cdots y_4)$ nor $(z_1 \cdots z_4)$ vanish or become infinite simultaneously for $x = a$. Then either the system $(\eta_1 \cdots \eta_4)$ or $(\zeta_1 \cdots \zeta_4)$ or both will become simultaneously zero or infinite for $x = a$.

Similar remarks may be made about a transformation of the invariant of weight four, θ_4. This vanishes for those values of x which correspond to generators, whose two intersections with the flecnode curve coincide. If a transformation of the independent variable be made

$$\xi = \xi(x),$$

we have

$$\xi'(x)^4 \theta_4(\xi) = \theta_4(x).$$

Therefore, if in an interval in which θ_4 does not vanish, we put

$$\xi = \int \sqrt[4]{\theta_4(x)}\, dx,$$

we shall have

$$\theta_4(\xi) = 1,$$

but clearly this is not legitimate for values of x for which $\theta_4 = 0$.

A similar transformation may be applied to any invariant which does not vanish identically.

§ 6. *The covariant* C_3.

We may write the semi-covariants E and N of § 2 as follows:

$$(52)\qquad E = \begin{vmatrix} \frac{1}{2}(v_{11} - v_{22})y + v_{12}z, & y \\ v_{21}y - \frac{1}{2}(v_{11} - v_{22})z, & z \end{vmatrix},$$

$$N = \begin{vmatrix} (u_{11} - u_{22})\rho + 2u_{12}\sigma, & y \\ 2u_{21}\rho - (u_{11} - u_{22})\sigma, & z \end{vmatrix}.$$

Therefore

$$(53)\qquad C_3 = E + 2N = \begin{vmatrix} 2(u_{11} - u_{22})\rho + 4u_{12}\sigma + \frac{1}{2}(v_{11} - v_{22})y + v_{12}z, & y \\ 4u_{21}\rho - 2(u_{11} - u_{22})\sigma + v_{21}y - \frac{1}{2}(v_{11} - v_{22})z, & z \end{vmatrix}.$$

If we take as curves C_y and C_z the two branches (assumed distinct), of the flecnode curve, we shall have $u_{12} = u_{21} = 0$, and therefore more simply

$$(54)\qquad C_3 = \begin{vmatrix} 2(u_{11} - u_{22})\rho + (u'_{11} - u'_{22})y - p_{12}(u_{11} - u_{22})z, & y \\ -2(u_{11} - u_{22})\sigma + p_{21}(u_{11} - u_{22})y - (u'_{11} - u'_{22})z, & z \end{vmatrix},$$

for (*Invariants*, equations (32)), if $u_{12} = u_{21} = 0$,

$$(55)\qquad v_{11} - v_{22} = 2(u'_{11} - u'_{22}),\quad v_{12} = -p_{12}(u_{11} - u_{22}),\quad v_{21} = +p_{21}(u_{11} - u_{22}).$$

Moreover θ_4 reduces to $(u_{11} - u_{22})^2$. If, then, we put

$$\xi = \int \sqrt{u_{11} - u_{22}}\, dx,$$

and if we introduce ξ as the independent variable in place of x, $u_{11} - u_{22}$ will be a constant, and, therefore,

$$u'_{11} - u'_{22} = 0.$$

These transformations are legitimate simultaneously for a region of the surface, which includes in its interior no generator along which the osculating hyperboloid hyperosculates the surface.

C_3 is thus simplified to

$$C_3 = (u_{11} - u_{22}) \begin{vmatrix} 2\rho - p_{12}z, & y \\ -2\sigma + p_{21}y, & z \end{vmatrix}. \tag{56}$$

Let us put

$$\alpha = 2\rho - p_{12}z, \qquad \beta = -2\sigma + p_{21}y, \tag{57}$$

and interpret these quantities geometrically.

The lines P_yP_z, $P_\rho P_\sigma$, P_yP_ρ and P_zP_σ form a skew quadilateral. P_yP_z is a generator of S, P_y and P_z being its intersection with the flecnode curve of S. $P_\rho P_\sigma$ is the generator of S' corresponding to P_yP_z, so that P_yP_z and $P_\rho P_\sigma$ are generators of the first kind and P_yP_ρ and P_zP_σ generators of the second kind of the hyperboloid osculating S along P_yP_z. Complete the tetrahedron determined by these four points by joining P_yP_σ and P_zP_ρ. Then α_k and β_k are clearly the coördinates of two points P_α and P_β on P_zP_ρ and P_yP_σ respectively. *These points are therefore reciprocal poles of each other with respect to the hyperboloid osculating S along P_yP_z.*

From the equations defining ρ and σ, we have

$$2y' = \rho - p_{11}y - p_{12}z,$$
$$2z' = \sigma - p_{21}y - p_{22}z,$$

and these are obviously the coördinates of two points $P_{y'}$ and $P_{z'}$ on the tangents to C_y and C_z at P_y and P_z respectively. Moreover, since $u_{12} = u_{21} = 0$, they are points on the two tangents to the flecnode curve. We may assume $p_{11} = p_{22} = 0$, for transformation (51) will make p_{11} and p_{22} vanish, and such a transformation clearly does not affect the geometrical significance of any of the quantities involved.

Then the points $P_{y'}$ and $P_{z'}$ determined by

$$2y' = \rho - p_{12}z,$$
$$2z' = \sigma - p_{21}y, \tag{58}$$

are clearly the intersections of the tangents of the flecnode curve with $P_\rho P_\rho$ and P_yP_σ respectively.

But from these equations we see that we may write

$$2y' = \tfrac{1}{2}(\alpha - p_{12}z), \qquad \rho = \tfrac{1}{2}(\alpha + p_{12}z),$$
$$2z' = -\tfrac{1}{2}(\beta + p_{21}y), \qquad \sigma = -\tfrac{1}{2}(\beta - p_{21}y).$$

so that the points $P_{y'}$ and P_ρ are harmonic conjugates with respect to P_α and P_z on the line $P_z P_\rho$, and similarly the points $P_{z'}$, P_σ, P_β and P_y form a harmonic range on $P_y P_\sigma$.

We thus have, besides S', a third ruled surface S'' associated with S in the following manner: Let P_y and P_z be the two flecnodes, supposed distinct, on a given generator of S. At each of these points three important lines intersect, viz., the generator, the flecnode tangent, and the tangent to the flecnode curve. All of these are in the plane tangent to the surface S at that point. In this plane pencil we construct a fourth line, so that with respect to it and the generator, the other two lines shall be harmonic conjugates. This line intersects the line joining the other flecnode to P_ρ in the point P_α. The point P_β is constructed in a similar way from a pencil having its vertex at P_z. The line $P_\alpha P_\beta$ is the generator of the surface S'' which corresponds to the generators $P_y P_z$ of S and $P_\rho P_\sigma$ of S'.

If, instead of the particular surface S', another of the covariant surfaces of § 3 had been chosen, by considering

$$\rho + \lambda y, \qquad \sigma + \lambda z$$

instead of ρ and σ, P_α and P_β would merely be displaced along the lines $P_\alpha P_y$ and $P_\beta P_z$ respectively. We should always obtain three ruled surfaces closely associated with each other.

The covariant C_3 vanishes identically if the ruled surface is of the second order. If it were to vanish in any other case we should have

$$2(u_{11} - u_{22})\rho + 4u_{12}\sigma + \tfrac{1}{2}(v_{11} - v_{22})y + v_{12}z = \lambda y,$$
$$4u_{21}\rho - 2(u_{11} - u_{22})\sigma + v_{21}y - \tfrac{1}{2}(v_{11} - v_{22})z = \lambda z,$$

i. e., the four points P_y, P_z, P_ρ, P_σ would be coplanar or, in subcases, even collinear. In other words the osculating hyperboloid would be degenerate, and the surface would be developable. *Therefore, C_3 can vanish identically only if the surface is of the second order.*

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ON THE RANK, ORDER AND CLASS OF ALGEBRAIC MINIMUM CURVES*

BY

ARTHUR SULLIVAN GALE

The equations of all analytic minimum curves may be written in terms of a complex parameter s and an arbitrary analytic function $F(s)$. The curve will be algebraic when and only when $F(s)$ is an algebraic function, and then the rank, order and class of the curve can be easily expressed as the orders of three functions, $\psi(s)$, $\phi(s)$ and $\chi(s)$, (see § 1), of the general form

$$f[s, F(s), F'(s), F''(s)].$$

It is the purpose of this paper to give the developments in series of

$$\psi(s), \phi(s) \quad \text{and} \quad \chi(s)$$

from which the rank, order and class of the minimum curve corresponding to $F(s)$ may be determined if the developments of $F(s)$ be known in the vicinity of its poles, branch points and the point at infinity. Detailed application is then made to the case in which $F(s)$ is a two-valued function of its argument, the case in which $F(s)$ is a rational function having been treated by LIE.† Finally attention is called to the relation of this work to the theory of minimum surfaces.

§ 1. *Equations and properties of minimum curves.*‡

All minimum curves, except minimum lines, may be represented in the form

$$\Gamma: \quad \begin{cases} x = (1-s^2)F'''(s) + 2sF''(s) - 2F(s) \\ y = i(1+s^2)F'''(s) - 2isF''(s) + 2iF(s) \\ z = 2sF'''(s) - 2F''(s) \end{cases}$$

where $F(s)$ is an analytic function subject to the sole condition that $F''''(s) \not\equiv 0$; Γ is an algebraic curve when and only when $F(s)$ is an algebraic function; its osculating plane is

$$(1-s^2)x + i(1+s^2)y + 2sz + 4F(s) = 0. \tag{1}$$

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† Mathematische Annalen, vol. 14 (1879), pp. 368, 371, 372.

‡ These properties are to be found in essentially the same form in DARBOUX, *Théorie des Surfaces*, §§ 235, 198.

THEOREM. *The order of* Γ *is the order of the function*

$$\phi(s) = A[(1-s^2)F'''(s) + 2sF''(s) - 2F(s)]$$
$$+ iB[(1+s^2)F'''(s) - 2sF''(s) + 2F(s)] + 2C[sF'''(s) - F''(s)] + D.$$

For to every value $s = s_0$ such that $\phi(s_0) = 0$ corresponds a point of intersection of Γ with the plane of general position

$$Ax + By + Cz + D = 0. \tag{2}$$

THEOREM. *The rank of* Γ *is the order of*

$$\psi(s) = 2A[(1-s^2)F''(s) + 2sF(s)] + 2iB[(1+s^2)F''(s) - 2sF(s)]$$
$$+ 4C[sF''(s) - F(s)] + iA_1(1-s^2) - B_1(1+s^2) + 2iC_1 s.$$

For the vanishing of $\psi(s)$ is the condition that the line of general position

$$\begin{aligned} Bz - Cy &= A_1 \\ Cx - Az &= B_1 \qquad AA_1 + BB_1 + CC_1 = 0 \\ Ay - Bx &= C_1 \end{aligned} \tag{3}$$

will intersect a generator

$$(1-s^2)x + i(1+s^2)y + 2sz + 4F(s) = 0$$
$$-sx + isy + z + 2F'(s) = 0$$

of the envelope of (1).

THEOREM. *The class of* Γ *is the order of*

$$\chi(s) = (1-s^2)A + i(1+s^2)B + 2sC + 4F(s).$$

For the condition that the plane (1) shall pass through the point (A, B, C) is that $\chi(s) = 0$.

THEOREM. *If* Γ *be subjected to a displacement the transformed curve is given by the equations*

$$x' = (1-t^2)G''(t) + 2tG'(t) - 2G(t)$$
$$y' = i(1+t^2)G''(t) - 2itG'(t) + 2iG(t)$$
$$z' = 2tG''(t) - 2G'(t)$$

where

$$(a) \qquad t = s, \qquad G(t) = F(s) - \frac{\lambda}{4}(1-s^2) - \frac{i\mu}{4}(1+s^2) - \frac{\nu}{2}s$$

if the displacement be a translation whose components parallel to the axes are λ, μ, ν; *or*

$$(b) \qquad t = \frac{m_0 s - n}{n_0 s + m}, \qquad \delta = mm_0 + nn_0,$$
$$G(t) = \frac{(m_0 - n_0 t)^2}{\delta} F\left(\frac{mt+n}{m_0 - n_0 t}\right)$$

if the displacement be a rotation about the origin through the same angle as the complex sphere is rotated by the transformation

$$t = \frac{m_0 s - n}{n_0 s + m}.$$

The first of these transformations shows that if

$$F(s) = f(s) + \alpha s^2 + \beta s + \gamma$$

it may be reduced to $f(s)$ by translating Γ through a distance whose components are $2(\gamma - \alpha), -2i(\gamma + \alpha), 2\beta$. The second transformation shows that by properly orienting the curve Γ the function $F(s)$ may be chosen in such a way that $s = \infty$ will not be one of its branch points. We will always suppose that Γ is oriented in this manner.

§ 2. *Development of the functions* $\phi(s)$, $\psi(s)$ *and* $\chi(s)$.

When $F(s)$ is an algebraic function it may be developed in the vicinity of any point of the plane, $s = a$, in one of the forms

$$F(s) = (s - a)^{\frac{p}{q}} P(s - a)^{\frac{1}{q}}, \tag{4}$$

$$F(s) = (s - a)^p P(s - a), \tag{5}$$

$$F(s) = \left(\frac{1}{s}\right)^p P\left(\frac{1}{s}\right), \tag{6}$$

$$F(s) = \left(\frac{1}{s}\right)^{\frac{p}{q}} P\left(\frac{1}{s}\right)^{\frac{1}{q}} \tag{7}$$

where p and q are integers and $P(x)$ denotes a series of the form

$$P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots,$$

in which $\alpha_0 \neq 0$. After the convention of the preceding section we need not consider developments of the form (7). Assuming $F(s)$ to be successively of the forms (4), (5) and (6) we shall determine the corresponding developments of $\phi(s)$, $\psi(s)$ and $\chi(s)$ about $s = a$; we shall then be able to determine the various types of points which are poles of these functions and if we can determine all points of $F(s)$ of the types found we can then determine the orders of $\phi(s)$, $\psi(s)$ and $\chi(s)$ by summing the orders of their poles. *The developments of this section are applicable to any algebraic minimum curve corresponding to a given function* $F(s)$ *and hence may be applied to the determination of the order, rank and class of that curve.*

Consider $\psi(s)$ first. If we suppose that $F(s)$ has the form (4) I find that $\psi(s)$ will have the form,

$$\psi(s) = 2(s-a)^{\frac{p-q}{q}}\left\{\beta\alpha_0\frac{p}{q} + \beta\alpha_1\frac{p+1}{q}(s-a)^{\frac{1}{q}} + \beta\alpha_2\frac{p+2}{q}(s-a)^{\frac{2}{q}} + \cdots\right.$$
$$+\left[\beta\alpha_q\frac{p+q}{q} + \beta'\alpha_0\frac{p-q}{q}\right](s-a) + \left[\beta a_{q+1}\frac{p+q+1}{q}\right.$$
$$\left.\left. + \beta'\alpha_1\frac{p-q+1}{q}\right](s-a)^{\frac{q+1}{q}} + \cdots\right\} + i\left\{\beta_1 + \beta_1'(s-a) + \frac{\beta_1''}{2}(s-a)^2\right\},$$

where

$$\beta = A(1-a^2) + iB(1+a^2) + 2Ca,$$
$$\beta_1 = A_1(1-a^2) + iB_1(1+a^2) + 2C_1a,$$

and primes denote differentiation with respect to a. Notice that we can always suppose $\beta \neq 0$, as otherwise the line (3) would not be a line of general position but would belong to the linear complex defined by $\beta = 0$. Now $s = a$ can be a pole of $\psi(s)$ only when $p < q$. If $0 < p < q$, $s = a$ will be a pole of order $q - p$; while if $p = 0$, the order of the pole will be $q - i$ if

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{i-1} = 0, \qquad \alpha_i \neq 0, \qquad i < q,$$

but the point will not be a pole if $i \geqq q$. When $p < 0$ the order of the pole at $s = a$ is $|p| + q$.

If $F(s)$ has the form (5) we have only to interpret the above results for $q = 1$. Hence in this case $s = a$ will be a pole of $\psi(s)$ only when $p < 0$ and then the order of the pole is $|p| + 1$.

Finally, in the vicinity of $s = \infty$, if $F(s)$ has the form (6) I find that

$$\psi(s) = 2\left(\frac{1}{s}\right)^{p-1}\left\{\gamma\alpha_0(p+2) + [\gamma\alpha_1(p+3) - 2C\alpha_0(p+1)]\left(\frac{1}{s}\right)\right.$$
$$\left. + [\gamma\alpha_2(p+4) - \delta\alpha_0 p - 2C\alpha_1(p+2)]\left(\frac{1}{s}\right)^2 + \cdots\right\}$$
$$+ i\left(\frac{1}{s}\right)^{-2}\left\{-\gamma_1 + 2C_1\left(\frac{1}{s}\right) + \delta_1\left(\frac{1}{s}\right)^2\right\},$$

where

$$\gamma = A - iB, \qquad \delta = A + iB,$$
$$\gamma_1 = A_1 - iB_1, \qquad \delta_1 = A_1 + iB_1.$$

When $p \geqq 0$, $s = \infty$ will be a pole of $\psi(s)$ of order 2, for we can suppose that $\gamma_1 \neq 0$. When $p = -1$ or -2 the order of the pole is still 2, for we can further suppose in the first case that $2\gamma\alpha_0 - i\gamma_1 \neq 0$ or, in the second case, that $2\gamma\alpha_1 + 4C\alpha_0 - i\gamma_1 \neq 0$. Hence, when $p \geqq -2$ we see that $s = \infty$ is a pole of $\psi(s)$ of order 2 and when $\gamma < -2$ it will be a pole of order $|p| + 1$.

Consider now the corresponding developments of $\phi(s)$.* When $F(s)$ has the form (4) I find that

$$\phi(s) = D + (s-a)^{\frac{p-2q}{q}}\left\{\beta\alpha_0\frac{p}{q}\frac{p-q}{q} + \beta\alpha_1\frac{p+1}{q}\frac{p-q+1}{q}(s-a)^{\frac{1}{q}}\right.$$
$$+ \beta\alpha_2\frac{p+2}{q}\frac{p-q+2}{q}(s-a)^{\frac{2}{q}} + \cdots + \left[\beta\alpha_q\frac{p+q}{q}\frac{p}{q}\right.$$
$$\left.+ \beta'\frac{p}{q}\frac{p-2q}{q}\alpha_0\right](s-a) + \left[\beta\alpha_{q+1}\frac{p+q+1}{q}\right.$$
$$\left.\left.+ \beta'\alpha_1\frac{p+1}{q}\frac{p-2q+1}{q}\right](s-a)^{\frac{q+1}{q}} + \cdots\right\},$$

where β and β' have the same significance as above. Then $s=a$ cannot be a pole of $\phi(s)$ unless $p<2q$. When $q<p<2q$ it is a pole of order $2q-p$; while if $p=q$ the order of the pole is $q-i$ if

$$\alpha_1=\alpha_2=\cdots=\alpha_{i-1}=0, \qquad \alpha_i\neq 0, \qquad i<q;$$

but if $i\geqq q$ then $s=a$ will not be a pole of $\phi(s)$. Likewise, if $0<p<q$ the order of the pole at $s=a$ is $2q-p$; when $p=0$ the pole is of order $2q-j$ if

$$\alpha_1=\alpha_2=\cdots=\alpha_{j-1}=0, \qquad \alpha_j\neq 0, \qquad j<q,$$

or if

$$\alpha_1=\alpha_2=\cdots=\alpha_{q-1}=0, \qquad \alpha_q\lesseqgtr 0, \qquad \alpha_{q+1}=\alpha_{q+2}=\cdots=\alpha_{j-1}=0,$$
$$\alpha_j\neq 0, \qquad q<j<2q;$$

but if $j\geqq 2q$ then $s=a$ will cease to be a pole of $\phi(s)$. Finally, if $p<0$ there will always be a pole at $s=a$ of order $|p|+2q$.

Interpreting these results for $q=1$ we will have the development of $\phi(s)$ derived from the form (5) of $F(s)$. In this case $s=a$ will be a pole of $\phi(s)$ only when $p<0$; for if $p=1$ the first coefficient, and if $p=0$ the first two coefficients of the series will vanish. When $p<0$ there will be a pole of $\phi(s)$ at $s=a$ of order $|p|+2$.

Finally, if $F(s)$ is of the form (6) I find that

$$\phi(s) = D + \left(\frac{1}{s}\right)^p\left\{-\gamma\alpha_0(p+1)(p+2)\right.$$
$$+ [-\gamma\alpha_1(p+2)(p+3) + 2C\alpha_0 p(p+2)]\left(\frac{1}{s}\right)$$
$$\left.+ [-\gamma\alpha_2(p+3)(p+4) + \delta\alpha_0 p(p+1) + 2C\alpha_1(p+1)(p+2)]\left(\frac{1}{s}\right)^2 + \cdots\right\}$$

where $\qquad \gamma = A - iB \quad \text{and} \quad \delta = A + iB.$

* As the general values of the constants A, B and C will prevent the poles of the coefficients of A, B and C from destroying each other, it follows that the infinite points of Γ will be derived from those values of s giving rise to poles of $\phi(s)$.

As in the preceding case, $s = \infty$ cannot be a pole of $\phi(s)$ unless $p < -2$; for if $p = -1$ or -2 the coefficients of the powers of $1/s$ with negative exponents will vanish. Hence $s = \infty$ is a pole of $\phi(s)$ only when $p < -2$ in which case it is a pole of order $|p|$.

The various developments of $\chi(s)$ can be so easily written down that only the conclusions drawn from those developments need be stated. When $F(s)$ is of the form (4) or (5) $s = a$ will be a pole of $\chi(s)$ when and only when $p < 0$, the order of the pole being $|p|$ in each case. But if $F(s)$ has the form (6) $s = \infty$ will be a pole of $\chi(s)$ of order 2 unless $p < -2$ when the order is $|p|$.

Thus we see that the only values of s which can be poles of $\phi(s)$, $\psi(s)$ or $\chi(s)$ are the poles and branch points of $F(s)$ and the point at infinity. When the developments of $F(s)$ about these points are known the order, rank and class of the minimum curve corresponding to $F(s)$ can be easily determined.

§3. *Application to the case in which $F(s)$ is a rational function.**

For completeness we will apply the results of the preceding section to the case which LIE considered from a less general standpoint. The application is simplified by a suitable choice of axes. Notice first, however, that if a rational function of s be subjected to a transformation of the form

$$s = \frac{\alpha s' + \beta}{\gamma s' + \delta}$$

that the transformed function is a rational function in which *the degree of the numerator equals that of the denominator.* Hence by rotating the minimum curve Γ we can always reduce the rational function $F(s)$ to a rational function the degree of whose numerator exceeds that of the denominator by 2; and then by applying an appropriate translation to Γ we can reduce $F(s)$ to a rational function whose numerator is of lower degree than its denominator. $F(s)$ has no branch points and its only poles are now at the zeros of the denominator of which we will suppose there are m of orders $m_1, m_2, \cdots, m_m$ respectively.

The results of §2 show that the order of $\psi(s)$, i. e., the *rank* of Γ is

$$R = \sum_1^m m_i + m + 2;$$

that the order of $\phi(s)$, which gives the *order* of Γ, is

$$O = \sum_1^m m_i + 2m,$$

*In geometrical language, the osculating plane of Γ envelops a developable containing the imaginary circle at infinity as a single line. In general, the circle at infinity will be an n-fold line on that developable if $F(s)$ is n-valued. Compare DARBOUX, l. c., §235.

and that the *class* of Γ, or the order of $\chi(s)$, is

$$C = \sum_1^m m_i + 2.$$

The simplest of these curves corresponds to the hypothesis $m = m_1 = 1$ or $F(s) = 1/(s-a)$. The rank of Γ is 4, its order 3 and its class 3. Numerous other examples are given by LIE in § 11 of the paper already cited. In the following sections we will extend LIE's work to the case in which $F(s)$ is two-valued.

§ 4. *Normal form of $F(s)$ when $F(s)$ is two-valued.*

The application of the results of § 2 to the case in which $F(s)$ is a two-valued function is materially simplified by a suitable orientation of the curve Γ. When $F(s)$ is two-valued it may be written in the form *

$$F(s) = R_1(s) + R_2(s)\sqrt{P(s)},$$

where $R_1(s)$ and $R_2(s)$ are rational functions and $P(s)$ a polynomial without multiple roots of degree λ. By rotating Γ about the origin $F(s)$ may be thrown into the form

$$F(s) = \frac{(m_0 - n_0 s)^2}{\delta}\left[\frac{P_1(s)}{P_2(s)} + \frac{P_3(s)}{P_4(s)}\frac{\sqrt{\prod_1^\lambda (s-e_i)}}{(m_0 - n_0 s)^{\lambda/2}}\right]$$

$$= \frac{P_1 P_4 (m_0 - n_0 s)^{\frac{\lambda}{2}} + P_2 P_3 \sqrt{\prod_1^\lambda (s-e_i)}}{\delta P_2 P_4 (m_0 - n_0 s)^{(\lambda-4)/2}},$$

where the P_i are polynomials such that the degrees of P_1 and P_3 are equal, respectively, to those of P_2 and P_4. When λ is even we may set $\lambda = 2p+2$ and when odd we may rationalize the denominator and set $\lambda + 1 = 2p+2$. Thus, in either case, if we neglect all factors common to the numerator and denominator and also neglect any constant factor, the effect of which will be to replace the given minimum curve by one similar to it, we may always write $F(s)$ in the form

$$F(s) = \frac{C\prod_1^l (s-a_i)^{l_i} + \prod_1^m (s-b_i)^{m_i}\sqrt{\prod_1^{2p+2}(s-e_i)}}{\prod_1^n (s-c_i)^{n_i}} \tag{8}$$

where

$$\sum_1^l l_i = \sum_1^n n_i + 2$$

* Compare LIE, Archiv for Mathematik og Naturvidenskab, vol. 2 (1877), p. 195.

$$\sum_1^n n_i = \sum_1^m m_i + p - 1.$$

By a suitable translation of Γ the first of these conditions may be replaced by the condition

$$\sum_1^l l_i < \sum_1^n n_i.$$

In the following sections we will suppose that $F(s)$ has the form (8) with the conditions

$$\sum_1^l l_i < \sum_1^n n_i = \sum_1^m m_i + p - 1. \tag{9}$$

Thus if p and $\sum m_i$ are given the value of $\sum n_i$ is determined and that of $\sum l_i$ is known as fully as proves necessary for our purposes. As the transformations to which $F(s)$ has been subjected do not introduce any negative exponents the case $\sum m_i = p = 0$, and hence $\sum n_i = -1$, is to be excluded although it is easily seen to be identical with the case $\sum n_i = p = 0$, $\sum m_i = 1$.

§ 5. *Development of $F(s)$ about its poles, branch points and the point at infinity.*

We have now to consider the various possible developments of (8) in the vicinity of those points which may be poles of any of the three functions whose orders we wish to compute. At first let the points a_i, b_i, c_i and e_i be all distinct. It is unnecessary to consider the developments of $F(s)$ about $s = a_i$ and $s = b_i$.

About any branch point $s = e_i$ we have

$$F(s) = \alpha_0 + \alpha_1 (s - e_i)^{1/2} + \alpha_2 (s - e_i) + \cdots$$

where certainly

$$\alpha_0, \alpha_1 \neq 0.$$

In the vicinity of $s = \infty$ we have

$$F(s) = \frac{Cs^{\Sigma l_i} \prod_1^l \left(1 - \frac{a_i}{s}\right)^{l_i} + s^{\Sigma m_i + p + 1} \prod_1^m \left(1 - \frac{b_i}{s}\right)^{m_i} \sqrt{\prod_1^{2p+2} \left(1 - \frac{e_i}{s}\right)}}{s^{\Sigma n_i} \prod_1^n \left(1 - \frac{e_i}{s}\right)^{n_i}}$$

$$= \left(\frac{1}{s}\right)^{-2} \left\{ \alpha_0 + \alpha_1 \left(\frac{1}{s}\right) + \cdots \right\}, \qquad \alpha_0 \neq 0$$

since from (9)

$$\sum m_i + p + 1 = \sum n_i + 2 > \sum l_i$$

and the development commences with terms of the same order in each sheet of the Riemann's surface of two sheets on which $F(s)$ is rational.

In the vicinity of $s = c_i$ we have

$$(10) \qquad F(s) = (s - c_i)^{-n_i}\{\alpha_0 + \alpha_1(s - c_i) + \cdots\}, \qquad \alpha_0 \neq 0$$

and the exponent of the first factor will be the same for each sheet of the Riemann's surface. It may happen, however, that the developments of

$$P = C\prod_1^l (s - a_i)^{l_i} \quad \text{and} \quad Q = \prod_1^m (s - b_i)^{m_i}\sqrt{\prod_1^{2p+2}(s - e_i)}$$

in the vicinity of some of these points, say the r points c_i, $i = n - r + 1, \cdots, n$, may be the same for a number of terms. Then in one sheet, say the upper one, the order of the pole of $F(s)$ at c_i will be less than n_i, and, in fact, the point may cease to be a pole in that sheet. In order that the first $r_i + 1$ terms of these developments be identical it is necessary and sufficient that

$$(11) \qquad \begin{cases} [P]_{s=c_i} = [Q]_{s=c_i}, \\ \dfrac{d}{ds}[P]_{s=c_i} = \dfrac{d}{ds}[Q]_{s=c_i}, \\ \dfrac{d^{r_i}}{ds^{r_i}}[P]_{s=c_i} = \dfrac{d^{r_i}}{ds^{r_i}}[Q]_{s=c_i}. \end{cases}$$

When these equations are satisfied $F(s)$ will have the form (10) about $s = c_i$ when s is in the lower sheet; but in the upper sheet its development will be of the form

$$(12) \qquad F(s) = (s - c_i)^{-n_i + r_i + 1}\{\alpha_0 + \alpha_1(s - c_i) + \cdots\}, \qquad \alpha_0 \neq 0.$$

We need never consider $r_i > n_i - 1$ as $r_i = n_i - 1$ is sufficient to make $s = c_i$ cease to be a pole of $F(s)$ and hence, also, cease to be a pole of $\phi(s)$, $\psi(s)$ or $\chi(s)$.

It is now necessary to examine all the possible cases in which the points a_i, b_i, c_i and e_i are not distinct. If we denote by an equality of the form $a_i = b_i$ the fact that one of the points $s = a_i$ is coincident with some one of the points $s = b_i$ the possible cases are:

1) $a_i = b_i$,
2) $a_i = c_i$,
3) $b_i = c_i$,
4) $a_i = e_i$,
5) $b_i = e_i$,
6) $c_i = e_i$,
7) $a_i = b_i = c_i$,
8) $a_i = b_i = e_i$,
9) $a_i = c_i = e_i$,
10) $b_i = c_i = e_i$,
11) $a_i = b_i = c_i = e_i$.

Of these we may at once exclude cases 7 and 11 from consideration. Case 1 may be neglected as it will not affect the developments of $F(s)$ in the vicinity of its poles, branch points and the point at infinity. Cases 2 and 3 have no effect on the developments of the form (10) except that condition (11) cannot be satisfied by such points. Cases 6 and 10 will give developments of the same form and so may be treated together. In the remaining cases $F(s)$ will asume developments of distinct forms.

For convenience of reference the developments needed in the following section are given in the accompanying table. The first column characterizes the point about which $F(s)$ has the form given in the second column where $P(x)$ denotes the series

$$P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots, \qquad (\alpha_0 \neq 0).$$

In the third column are remarks of an essential nature concerning the series in the second column.

e_i	$P(s-e_i)^{\frac{1}{2}}$	$\alpha_1 \neq 0$
$c_i,\ i \leqq n-r$	$(s-c_i)^{-n_i}P(s-c_i)$	in each sheet.
$c_i,\ i > n-r$	$(s-c_i)^{-n_i+r_i+1}P(s-c_i)$ $(s-c_i)^{-n_i}P(s-c_i)$	in one sheet. in other sheet.
$a_i = e_i$	$(s-e_i)^{\frac{1}{2}}P(s-e_i)^{\frac{1}{2}}$	
$b_i = e_i$	$P(s-e_i)^{\frac{1}{2}}$	$\alpha_1 = 0$ $\alpha_3 \neq 0$ if $m_i = 1$ $\alpha_3 = 0$ " $m_i > 1$
$c_i = e_i$	$(s-e_i)^{-\frac{2n_i}{2}}P(s-e_i)^{\frac{1}{2}}$	
$a_i = b_i = e_i$	$(s-e_i)^{\frac{\epsilon}{2}}P(s-e_i)^{\frac{1}{2}}$	$\epsilon = 2,\ \alpha_1 \neq 0$ if $m_i = 1,\ l_i = 1$ $\epsilon = 3$ " $m_i = 1,\ l_i > 1$ $\epsilon = 2,\ \alpha_1 = 0$ " $m_i > 1,\ l_i = 1$ $\epsilon > 3$ " $m_i > 1,\ l_i > 1$
$a_i = c_i = e_i$	$(s-e_i)^{-\frac{2n_i-1}{2}}P(s-e_i)^{\frac{1}{2}}$	
∞	$\left(\frac{1}{s}\right)^{-2}P\left(\frac{1}{s}\right)$	in each sheet.

§ 6. *The rank, order and class of* Γ *when* $F(s)$ *is two-valued.*

The rank, order and class of the minimum curve Γ corresponding to the two-valued function $F(s)$ can now be obtained by combining the results of § 2 and § 5. Notice first that the points $a_i = e_i$ do not need to be distinguished from the points e_i; for both give rise to simple poles of $\psi(s)$ and poles of $\phi(s)$ of order 3. In like manner the points $a_i = b_i = e_i$ may be counted with the points $b_i = e_i$ since neither are poles of $\psi(s)$ and they are simple poles of $\phi(s)$ when and only when $m_i = 1$, ceasing to be poles if $m_i > 1$. But one further point needs special mention, viz., the r points (r may be zero) $s = c_i$ which satisfy the conditions (11). In one sheet of the Riemann's surface $s = c_i$ will be a pole of $\psi(s)$ of order $n_i + 1$ and of $\phi(s)$ of order $n_i + 2$; while in the other sheet the order of the pole of $\psi(s)$, if the point is a pole, will be $n_i + 1 - \rho_i$ where

$$\rho_i = \begin{cases} 0 & \text{if conditions (11) are not satisfied,} \\ r_i + 1 & \text{“} \quad r_i < n_i - 1, \\ n_i + 1 & \text{“} \quad r_i \geqq n_i - 1, \end{cases} \tag{13}$$

the point ceasing to be a pole when ρ_i assumes its maximum value; and the order of the pole $\phi(s)$ at $s = c_i$, if the point is a pole, will be $n_i + 2 - \sigma$ where

$$\sigma_i = \begin{cases} 0 & \text{if conditions (11) are not satisfied,} \\ r_i + 1 & \text{“} \quad r_i < n_i - 1, \\ n_i + 2 & \text{“} \quad r_i \geqq n_i - 1, \end{cases}$$

the point ceasing to be a pole when σ_i takes on its maximum value.

The following table is convenient in summing the orders of $\psi(s)$ and $\phi(s)$. The first column gives the number of points of $F(s)$ of the sort characterized in the second column; the numbers defined in this column are evidently subject to the conditions

$$\mu + \nu + r \leqq n,$$

$$\lambda + \mu + \nu \leqq 2p + 2,$$

$$\lambda_1 + \lambda_2 = \lambda.$$

In the third column is given the order of the pole of $\psi(s)$ at the corresponding point in the second column; and in the fourth column in the sum of the orders over all such points. The fifth and sixth columns bear the same relation to $\phi(s)$ as the third and fourth do to $\psi(s)$. In counting the number of points at the poles $s = c_i$ and the point at infinity the number is doubled to account for both sheets of the Riemann's surface on which $F(s)$ is rational.

$\lambda \begin{cases} \lambda_1, m_i=1 \\ \lambda_2, m_i>1 \end{cases}$	$b_i=e_i$	no pole		$\begin{cases} 1 \\ \text{no pole} \end{cases}$	λ_1
ν	$e_i=e_i$	$2n_i+2$	$2\sum_1^{\nu} n_i+2\nu$	$2n_i+4$	$2\sum_1^{\nu} n_i+4\nu$
μ	$a_i=e_i=e_i$	$2n_i+1$	$2\sum_{\nu+1}^{\nu+\mu} n_i+\mu$	$2n_i+3$	$2\sum_{\nu+1}^{\nu+\mu} n_i+3\mu$
$2p+2$ $-(\lambda+\mu+\nu)$	e_i	1	$2p+2-\lambda-\mu-\nu$	3	$6p+6-3\lambda$ $-3\mu-3\nu$
$2n$ $-2(\mu+\nu+r)$	$e_i, i \leqq n-r$	n_i+1	$2\sum_{\nu+\mu+1}^{n-r} n_i+2n$ $-2\mu-2\nu-2r$	n_i+2	$2\sum_{\nu+\mu+1}^{n-r} n_i+4n$ $-4\mu-4\nu-4r$
$2r$	$e_i, i>n-r$	$\begin{cases} n_i+1 \\ n_i+1-p_i \end{cases}$	$2\sum_{n-r+1}^{n} n_i+2r$ $-\sum_{n-r+1}^{n} \rho_i$	$\begin{cases} n_i+2 \\ n_i+2-\sigma_i \end{cases}$	$2\sum_{n-r+1}^{n} n_i+4r$ $-\sum_{n-r+1}^{n} \sigma_i$
2	∞	2	4	no pole	

Summing the fourth and sixth columns we have the

THEOREM.—*The rank of the minimum curve* Γ *corresponding to*

$$F(s)=\frac{C\prod_1^{l}(s-a_i)^{l_i}+\prod_1^{m}(s-b_i)^{m_i}\sqrt{\prod_1^{2p+2}(s-e_i)}}{\prod_1^{n}(s-c_i)^{n_i}},$$

$$\sum_1^{l} l_i<\sum_1^{n} n_i=\sum_1^{m} m_i+p-1$$

is

$$R=2p+2n+2\sum_1^{n} n_i+6-\lambda-2\mu-\nu-\sum_{n-r+1}^{n}\rho_i$$

and the order of Γ *is*

$$O=6p+4n+2\sum_1^{n} n_i+6-2\lambda-\lambda_2-4\mu-3\nu-\sum_{n-r+1}^{n}\sigma_i.$$

If, however, $F(s)$ *is defined by a binomial equation we must have*

$$\nu=\rho_i=\sigma_i=0$$

and μ *will denote the number of points* $c_i=e_i$.

The last part of the theorem follows by comparing the developments of

$$F(s)=\frac{\prod_1^m (s-b_i)^{m_i}\sqrt{\prod_1^{2p+2}(s-e_i)}}{\prod_1^n (s-c_i)^{n_i}}, \qquad \sum_1^n n_i=\sum_1^m m_i+p-1$$

with those given in § 5; for μ and ν denote essentially the numbers of branch points about which the developments of $F(s)$ in series start with terms whose exponents are respectively $-(2n_i-1)/2$ and $-2n_i/2$; hence μ denotes in this case the number of points $c_i=e_i$ and $\nu=0$.

The application of that part of § 2 which deals with $\chi(s)$ to the developments in § 5 gives easily the result that

The class of Γ *is*

$$C=2\sum_1^n n_i-\mu+4-\sum_{n-r+1}^n \tau_i$$

where

$$\tau_i=\begin{cases} 0 & \text{if conditions (11) are not satisfied,}\\ r_i+1 & \text{“ } r_i<n_i-1,\\ n_i & \text{“ } r_i\geqq n_i-1. \end{cases}$$

§ 7. *Examples.*

To illustrate the results of the preceding section we will consider the minimum curves of lowest rank for which $F(s)$ is two-valued. All curves of given rank R may be obtained by finding all integral solutions of the inequalities

$$(14)\qquad \begin{cases} 2p+2n+2\sum_1^n n_i+6-\lambda-2\mu-\nu-\sum_{n-r+1}^n \rho_i=R\\ \lambda+\mu+\nu\leqq 2p+2\\ \mu+\nu+r\leqq n \end{cases}$$

consistent with (9) and (13). It is not difficult to see that there are no solutions of these inequalities unless $R>4$. For we have

$$\lambda+\mu+\nu\leqq 2p+2,$$

$$\mu+\sum_{n-r+1}^n \rho_i\leqq\sum_{n-r+1}^n n_i+r+\mu\leqq\sum_1^n n_i+n,$$

hence, adding,

$$\lambda+2\mu+\nu+\sum_{n-r+1}^n \rho_i\leqq 2p+n+\sum_1^n n_i+2.$$

Therefore

$$R \geqq n + \sum_1^n n_i + 4,$$

and hence $R > 4$ unless $n = \sum n_i = 0$; in the latter case

$$\mu = \nu = \rho_i = 0, \qquad \sum m_i = 1,$$

and hence $\lambda \leqq 1$ which gives $R \geqq 5$.

We will now consider the three cases in which $R = 5$, 6 or 7.

1. $R = 5$. The inequalities (14) admit of the single solution

$$p = n = n_i = \mu = \nu = \rho_i = 0, \qquad \lambda = 1,$$

which gives the function

$$F(s) = (s - e_1)\sqrt{(s - e_1)(s - e_2)}.$$

The corresponding minimum curve Γ is of order 4 and class 4. The equations of this curve, in two different positions, have been given by RICHMOND* in *rational* form. These positions of Γ are obtained by rotating Γ so that $F(s)$ is replaced by $s\sqrt{s}$ or $(s - e_1)\sqrt{(s - e_1)s}$; in the latter case RICHMOND's point of view gives a geometrical interpretation of the constant e_1.

2. $R = 6$. When $R = 6$ there are two solutions of (14). The first is

$$p = n = n_i = \mu = \nu = \rho_i = \lambda = 0.$$

This solution gives

$$F(s) = (s - b)\sqrt{(s - e_1)(s - e_2)},$$

which reduces to the preceding case if $b = e_1$ or e_2. Γ is of order 6 and class 4.

The second solution is

$$n = n_1 = 1, \qquad p = \mu = \nu = 0, \qquad \lambda = \rho_1 = 2.$$

so that

$$F(s) = \frac{C + (s - e_1)(s - e_2)\sqrt{(s - e_1)(s - e_2)}}{(s - c)},$$

$$C = (c - e_1)^{\frac{3}{2}}(c - e_2)^{\frac{3}{2}}.$$

The corresponding minimum curve is of order 5 and class 5.

The preceding curves have been given by LIE,† but his point of view did not enable him to give the functions $F(s)$ to which the curves correspond.

* *On the Simplest Algebraic Minimal Curves, and the Derived Real Minimal Surfaces*, Transactions of the Cambridge Philosophical Society, vol. 19 (1900), pp. 74, 75. In accordance with the theorem of SCHWARZ, Mathematische Abhandlungen, vol. 2, p. 22, the curves of rank 7 or less may be expressed *rationally* in terms of a parameter and hence we expect that there are no solutions of (14) for $R \leqq 7$ unless $p = 0$.

† l. c., p. 380.

3. $R = 7$. There are three solutions of (14) when $R = 7$. The first is

$$n = n_1 = 1, \quad p = \mu = \nu = 0, \quad \lambda = 1, \quad \rho_1 = 2,$$

which gives

$$F(s) = \frac{C + (s-b)(s-e_1)\sqrt{(s-e_1)(s-e_2)}}{s-c}.$$

$$C = (c-b)(c-e_1)^{\frac{1}{2}}(c-e_2)^{\frac{1}{2}}.$$

The order of Γ is 7 unless $b = e_1$ when the order reduces to 6. In either case the class is 5. If $b = e_2$ then R reduces to 6.

A second solution is

$$n = 1, \quad n_1 = 2, \quad p = \mu = \nu = 0, \quad \lambda = 2, \quad \rho_1 = 3,$$

so that

$$F(s) = \frac{C(s-a) + (s-b)(s-e_1)(s-e_2)\sqrt{(s-e_1)(s-e_2)}}{(s-c)^2},$$

$$C(c-a) = (c-b)(c-e_1)^{\frac{1}{2}}(c-e_2)^{\frac{1}{2}}$$

$$C\frac{d}{ds}[s-a]_{s=c} = \frac{d}{ds}[(s-b)(s-e_1)^{\frac{1}{2}}(s-e_2)^{\frac{1}{2}}]_{s=c},$$

The order of Γ is 6 unless $b = e_1$ or e_2 when the order is 5; its class is 6.

Finally, there is a solution for $F(s)$ defined by a binomial quadratic; viz.:

$$p = 0, \quad n = n_1 = \mu = \lambda = 1.$$

Hence $$F(s) = \frac{(s-b)(s-e_1)\sqrt{(s-e_1)(s-e_2)}}{(s-e_2)}.$$

The order of the corresponding minimum curve is 6 or if $b = e_1$, it is 5. The class is 5.

In like manner we can determine all minimum curves for which $F(s)$ is two-valued of a given rank, order or class.

The examples which have been given all correspond to solutions of (14) of the form $p = 0$, $n \leqq 1$, $n_1 \leqq 2$; it is not difficult, although somewhat tedious, to show that there are no other solutions of this form if $R \leqq 7$. That all possible solutions for $R \leqq 7$ have been given will follow if it is shown that there are no solutions of the three forms

1. $p = 0, \quad n = 1, \quad n_1 > 2,$
2. $p = 0, \quad n > 1,$ (hence, also, for $p > 0$, $n > 1$.)
3. $p > 0, \quad n \leqq 1.$

The proof in cases 1 and 2 follows at once from the fact that

$$R \geqq n + \sum_1^n n_i + 4$$

for in each case $n + \sum n_i > 3$. In case 3 notice that

$$\mu + \sum_{n-r+1}^{n} \rho_i \leqq \sum_1^n n_i + n$$

$$\mu + \nu \leqq n$$

and hence

$$2\mu + \nu + \sum \rho_i \leqq 2n + \sum n_1.$$

Also

$$\lambda \leqq m \leqq \sum m_i = \sum n_i - p + 1.$$

Hence

$$R \geqq 3p + 5 \geqq 8$$

if $p > 0$.

§ 8. *Application to minimum surfaces.*

In the paper already cited LIE has shown that on a minimum surface any two minimum curves of the same family are congruent and may be brought into coincidence by a translation. He then gives rules for the determination of the order and class of a minimum surface in terms of the rank and order of the minimum curves lying on the surface and the number of values the functions defining those curves take on for a given value of the argument. Hence the results of the preceding sections are immediately applicable to the determination of the order and class of algebraic minimum surfaces.

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ON THE DETERMINATION OF THE DISTANCE BETWEEN TWO POINTS IN SPACE OF n DIMENSIONS*

BY

H. F. BLICHFELDT

Introduction.

The investigations on the foundation of geometry of RIEMANN, HELMHOLTZ and LIE have as a basal notion the distance between two points of space, and the analytic representation of this distance plays an important rôle. A point in space is determined by means of three independent coördinates; and the distance between two points is a certain function of their six coördinates. Rigid bodies are used and the continuous motions of such, determined by the continuous variation of the coördinates of their points. This notion of three independent and continuous coördinates of a point is, in reality, not simple, as has been pointed out by LIE and others. In an article entitled *Ein Beitrag zur Mannigfaltigkeitslehre*, Borchhardt's Journal, vol. 84 (1877), G. CANTOR describes a space in which the coördinates of a point are neither independent nor continuous.

It does not seem easy, however, to determine the distance between two points of space if the coördinates of a point are neither independent nor continuous. It would not be evident, a priori, that certain loci (as spheres and circles) exist, defined by means of certain equations in the coördinates of their points. Such loci are used in the processes employed by HELMHOLTZ and LIE, and the question arises whether their axioms involving such loci and the motion of points in space, etc., should be modified or replaced by others.

One such new axiom, used in the following paper, refers to *distance-relations*, i. e., it is assumed that certain definite relations connect the mutual distances between any $n + 2$ different points in space of n dimensions.†

One notices that the assumption, used by HELMHOLTZ and LIE and their successors, that the distance between two points in space is a certain function of the

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† Such an axiom has been given by M. J. DE TILLY in his *Essai de géométrie analytique générale*, Mémoires couronnés et autres mémoires publiés par l'Académie Royale etc. de Belgique, vol. 47 (1892-93).

six coördinates of the two points, implies the existence of distance-relations; for between the mutual distances (28) of eight points we could obviously eliminate the 24 coördinates involved, giving us at least four general relations.

The following paper is an attempt to define the analytic form of the distance between two points by means of a set of axioms, based on the assumption that distance-relations exist. The continuity and independence of the coördinates used will not be assumed. A system of typical differential equations will be found — the same found by LIE — which are satisfied by the functions representing the distance.

Some remarks regarding the definitions given below may be in place here. It is not necessary for us to suppose that *space* is a continuum of points, as this phrase is generally understood. We may, for instance, select all the points in ordinary euclidean space whose cartesian coördinates are integers only. The *distances* need not form a continuous series of numbers.

In order to deal with known functions we shall at the outset confine ourselves to algebraic distance-relations, though a much wider range of relations may be treated in practically the same manner.

The analysis given does not discriminate between actual distance and any function of this distance. We shall, therefore, understand that the distance-relations are algebraic in the actual distances (as in euclidean space) or in certain transcendental functions of the distances (as in hyperbolic or elliptic space). If the latter is the case, such transcendental functions of the distances will be termed the distances themselves; and in the former case it is sometimes convenient to use the word *distance* for a certain algebraic function of the actual distance.

It must be borne in mind that the purpose of this paper is the discovery of the functions that may represent the distance between two points in a space defined by means of certain axioms given below, and not a discussion of the geometric properties of such a space. Certain properties of the functions referred to are found in § 2, and here nothing is assumed in regard to continuity of the coördinates involved. The distances may enjoy certain additional properties if the coördinates be *supposed* continuous and independent, and such properties are found (Theorems 5 and 6) which, in connection with the Theorems 1–4 form the bases for the succeeding paragraphs, in which the coördinates are throughout considered continuous and independent. *Infinitesimal transformations* are here introduced for the purpose of analysis, and have no reference to geometric transformations of points of the spaces considered.

§ 1. DEFINITIONS AND AXIOMS.

Art. 1. *Definitions.*

1. *Space* (of any number of dimensions) is a collection of an unlimited number of points.

2. To any pair of points of space corresponds a real number, in general finite, called the *distance* between the two points. If these be designated a and b, we shall denote the distance between them by $\overline{a,b}$, or $\overline{b,a}$.

3. For a certain integer m the mutual distances between any m points of a given space may, without exceptions, satisfy certain definite, algebraic relations. Such relations are then called *m-point distance-relations*, or simply *distance-relations*.

4. *Space of n dimensions* is a space for which $(n+2)$-point distance-relations (rational and integral) exist, but no such k-point distance-relations, if $k < n+2$. The symbol S_n means "space of n dimensions."

Art. 2. *Axioms.*

I. For any $n > 0$, a space of n dimensions exists.

II. If a finite number of points be cut out of a S_n the remaining points will still form a S_n.

III. A S_n contains always a S_{n-1}, if $n > 1$.

IV.* We shall assume that such m points (fixed) $1, 2, \cdots, m$, $(m \leqq n)$, of any given S_n—denoted by (A)—may be selected, and such a S_n, contained in (A), and denoted by (B), may be defined, that,

1°. The distance $\overline{a,b}$—a, b being any pair of points of (B)—can be expressed as a uniform, algebraic function of the $2m$ distances $\overline{1,a}, \overline{2,a}, \cdots, \overline{m,a}$; $\overline{1,b}, \overline{2,b}, \cdots, \overline{m,b}$, supposed algebraically independent of each other. (The functions considered can, for example, be expanded in a convergent power-series in the distances $\overline{1,a}, \cdots, \overline{m,b}$, as long as a, b remain within (B));

2°. For any S_n contained in (B), there is no algebraic equation satisfied by the distances $\overline{1,a}, \overline{2,a}, \cdots, \overline{m,a}$, a being any point of this S_n.

3°. Given an algebraic equation in the distances $\overline{1,2}, \overline{1,3}, \overline{2,3}, \cdots, \overline{n-1,n}$; the points $1, 2, 3, \cdots, n$ may be selected such that this equation is not satisfied.

Definition 5. The distances $\overline{1,t}, \overline{2,t}, \cdots, \overline{m,t}$ between any point t of (B) and the points $1, 2, \cdots, m$ shall be called the *coördinates* of the point t, and shall be denoted by $x_1^t, x_2^t, \cdots, x_m^t$, or simply by $x_1, x_2, \cdots, x_m$.

Thus, to every point t of (B) we have a set of m coördinates. It is not necessarily true, however, that a set of values arbitrarily assigned to the numbers $x_1^t, x_2^t, \cdots, x_m^t$ will make them coördinates of a point of (B), nor is it necessarily true that two points of (B) may be found whose coördinates differ by infinitesimals.

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* By means of the algebraic $(n+2)$-point distance-relations, one, at least, of the $\frac{1}{2}(n+2)(n+1)$ distances between any $n+2$ points of a given S_n is an algebraic function of the remaining distances. The nature of this function, when some of the $n+2$ points occupy definite positions, is more particularly defined by means of this Axiom (IV).

By Axiom IV, for points a, b of (B) the distance $\overline{a,b}$ is a uniform algebraic function of the $2m$ coördinates of the points a and b, say

$$\overline{a,b} = \phi(x_1^a, x_2^a, \cdots, x_m^a; x_1^b, x_2^b, \cdots, x_m^b). \tag{1}$$

§ 2. Theorems.

Art 3. Let $a, b, c, \cdots$ be any $n+2$ different points of (B), and let

$$\psi(\overline{a,b}, \overline{a,c}, \overline{b,c}, \cdots) = 0 \tag{2}$$

be one of the $(n+2)$-point distance-relations holding throughout (A); the function ψ is a rational integral function of its $\frac{1}{2}(n+2)(n+1)$ arguments.

1. *The $\frac{1}{2}(n+2)(n+1)$ distances $\overline{a,b}$, $\overline{a,c}$, $\overline{b,c}$, $\cdots$, written in the coördinates of the points $a, b, c, \cdots$, must satisfy identically the equation $\psi = 0$.*

Suppose, after substitution in $\psi = 0$ of the distances $\overline{a,b}$, etc., in terms of the coördinates x_1^a, x_1^b, etc., there results an equation $\rho = 0$ connecting the $m(n+2)$ coördinates considered.

By taking for the point a of (B) a definite point, so that the coördinates $x_1^a, x_2^a, \cdots, x_m^a$ are definite constants, this algebraic equation becomes an identity or an equation in the $m(n+1)$ coördinates of any $n+1$ points $b, c, \cdots$ of the S_n obtained by cutting out of (B) the point a. In case it becomes an identity if the $m(n+1)$ coördinates $x_1^b, x_2^b, \cdots, x_m^b, x_1^c, \cdots$ be considered arbitrary variables, and the point a selected be any point whatever of (B), we can give to the coördinates $x_1^b, x_2^b, \cdots, x_m^b; x_1^c, \cdots$ in $\rho = 0$ arbitrary constant values, and will thereby obtain an equation in the coördinates $x_1^a, x_2^a, \cdots, x_m^a$ true for every point a of (B). Thus, from an equation $\rho = 0$ connecting the coördinates of $n+2$ points of (B) will result an equation connecting the coördinates of less than $n+2$ points of (B), or of a S_n contained in (B). The process of reduction may be carried on till we have obtained an algebraic equation $\sigma = 0$ in the m coördinates of any point of a S_n contained in (B). But, by Axiom IV, no such equation can exist. The theorem is therefore true.

2. *The number m used in Axiom IV must be equal to n; i. e., the n distances between the points $1, 2, \cdots, n$ of (A) and any point of (B) are algebraically independent of each other.*

According to Axiom III, the space (B) contains a S_{n-1}, which we shall call (C). Let $\chi(\overline{a,b}, \overline{a,c}, \overline{b,c}, \cdots) = 0$ be one of the algebraic relations connecting the mutual distances of $n+1$ points $a, b, c, \cdots$ of (C). This equation is not identically satisfied if we substitute for $\overline{a,b}$, $\overline{a,c}$, $\cdots$ their values in terms of the $m(n+1)$ coördinates $x_1^a, \cdots$, or the equation $\chi = 0$ would hold throughout (B). Hence, an algebraic equation $\rho' = 0$ results, from which, as from the

equation $\rho = 0$ above, we find an algebraic equation connecting the coördinates of any point of (C) or of a S_{n-1} contained in (C).

According to Axiom IV, we can select a new system of coördinates for this S_{n-1}, say $y_1, y_2, \cdots, y_k$, which represent distances between k definite points of (C) and any point of a S_{n-1} (D) contained in (C). All these points lying in (C), their distances are functions of the former system of coördinates, and the coördinates $y_1', y_2', \cdots, y_k'$ of any point t of (D) are given in terms of definite constants and of the coördinates $x_1', x_2', \cdots, x_m'$. At most $m-1$ of these are algebraically independent for points of (C), as we have seen, and therefore $k \leqq m-1$, since the numbers $y_1', y_2', \cdots, y_k'$ are algebraically independent.

The space (D) contains a S_{n-2}, say (E). For this space we have a system of coördinates $z_1, z_2, \cdots, z_l$. We find $l \leqq k-1$.

Proceeding thus, we get a series of spaces of dimensions

$$n, n-1, n-2, \cdots, 1,$$

with systems of coördinates of numbers

$$m, k, l, \cdots, v,$$

with the relations

$$n \geqq m > k > l > \cdots > v > 0.$$

The number v (for space of one dimension) is 1 at least, as all distances in that space would otherwise be the same constant. This is impossible for a space of one dimension, by Definition 4. It follows that

$$v = 1, \cdots, l = n-2, k = n-1, m = n.$$

3. *In any S_n, but one $(n+2)$-point algebraic distance-relation exists.*

For, if we had more than one such, we would have two or more relations connecting the distances between the points $1, 2, \cdots, n, a, b$ used in Axiom IV, from which would follow one or more algebraic relations connecting the distances $\overline{1,2}, \overline{1,3}, \cdots, \overline{n-1,n}$ and the $2n$ coördinates of the two points a, b, which could not be free from the latter by Axiom IV, part 3°, the points $1, 2, 3, \cdots, n-1, n$ being properly chosen. Such relations would, as we have seen, lead to relations connecting the n coördinates of any one point of a S_n. Hence the theorem.

4. *The function*

$$\overline{a,b} = \phi(x_1^a, x_2^a, \cdots, x_n^a; x_1^b, x_2^b, \cdots, x_n^b)$$

is unaltered if all the coördinates of each point be interchanged with the corresponding coördinates of the other.

The contrary hypothesis would, namely, lead to an algebraic equation connecting the coördinates of any one point, following from

$$\overline{a,b} = \overline{b,a} = \phi(x_1^a, \cdots, x_n^a;\ x_1^b, \cdots, x_n^b) = \phi(x_1^b, \cdots, x_n^b;\ x_1^a, \cdots, x_n^a).$$

5. *If* $1', 2', 3', \cdots, n', a$ *are points of* (B), *then are the distances*

$$\overline{1',a},\ \overline{2',a},\ \cdots,\ \overline{n',a}$$

independent of each other with respect to the coördinates $x_1^a, x_2^a, \cdots, x_n^a$; *these, as well as the coördinates* $x_1^{1'}, \cdots, x_n^{1'}; \cdots; x_1^{n'}, \cdots, x_n^{n'}$ *being considered independent variables.*

We may, according to Axiom IV, choose new coördinates $y_1, y_2, \cdots, y_n$ for a S_n, which we shall call (G), contained in (B). These coördinates represent the distances from n definite points, say $1', 2', \cdots, n'$, of (B) and any given point of (G). The points $1', 2', \cdots, n'$ may, moreover, be chosen so that the distances $\overline{1',2'}, \overline{1',3'}, \overline{2',3'}, \cdots, \overline{(n-1)',n'}$, do not satisfy a given algebraic equation, $\sigma = 0$. This equation we shall determine in the following manner:

Let, for the moment, $1', 2', \cdots, n'$ be arbitrary points of (B), and let $a, b, c, \cdots$ be other $\frac{1}{2}n(n+1)+1$ points of (B). Suppose Theorem 5 not true; then we would have one or more algebraical identities of the form

$$\zeta(\overline{1',a}, \overline{2',a}, \cdots, \overline{n',a};\ x_1^{1'}, x_2^{1'}, \cdots, x_n^{1'};\ x_1^{2'}, \cdots, x_n^{2'};\ \cdots;\ x_1^{n'}, x_2^{n'}, \cdots, x_n^{n'}) = 0,$$

and similar equations for the points $b, c, \cdots$. Between these and the $\frac{1}{2}n(n-1)$ equations

$$\overline{1',2'} = \phi(x_1^{1'}, x_2^{1'}, \cdots, x_n^{1'};\ x_1^{2'}, x_2^{2'}, \cdots, x_n^{2'}),\ \overline{1',3'} = \text{etc.}, \cdots, \overline{(n-1)',n'} = \text{etc.},$$

we can eliminate the n^2 coördinates $x_1^{1'}, \cdots, x_n^{n'}$ involved, thereby obtaining at least one algebraic equation $\tau = 0$ connecting the distances

$$\overline{1',2'},\ \overline{1',3'},\ \cdots,\ \overline{(n-1)',n'};\ \overline{1',a},\ \cdots,\ \overline{n',a};\ \overline{1',b},\ \cdots,\ \overline{n',b},\ \text{etc.}$$

Let us, in this equation, substitute arbitrary, but definite constants for the distances $\overline{1',a}, \overline{2',a}, \cdots, \overline{n',a}; \overline{1',b}, \overline{2',b}$, etc. We get an equation in the distances $\overline{1',2'}, \overline{1',3'}, \overline{2',3'}, \cdots, \overline{(n-1)',n'}$, which is not satisfied in general, as the points $1', 2', \cdots, n'$ are taken from a S_n (Definition 4), but might be satisfied for special points of (B). This is the equation we shall call $\sigma = 0$, and the points $1', 2', \cdots, n'$ shall be chosen so that it is not satisfied. If so, the equation $\tau = 0$ cannot be an identity, but is an actual equation in the distances $\overline{1',a}, \overline{2',a}, \cdots, \overline{n',a}; \overline{1',b}$, etc., which are now the coördinates $y_1^a, \cdots, y_n^a$, etc., of the points $a, b, c, \cdots$, of (G). But, the existence of an algebraic equation connecting the coördinates of any given number of arbitrary points of (G) leads to an algebraic equation connecting the coördinates of any point of a S_n contained in (G). Such an equation cannot exist by Theorem 2. Hence the Theorem 5.

6. *The distance* $\overline{a,b} = \phi(x_1^a, \cdots, x_n^a; x_1^b, \cdots, x_n^b)$ *does not satisfy a linear partial differential equation of the form*

$$X(\overline{a,b}) \equiv A_1 \frac{\partial(\overline{a,b})}{\partial x_1^a} + A_2 \frac{\partial(\overline{a,b})}{\partial x_2^a} + \cdots + A_n \frac{\partial(\overline{a,b})}{\partial x_n^a} = 0,$$

where the coefficients $A_1, A_2, \cdots, A_n$ *are functions of* $x_1^a, x_2^a, \cdots, x_n^a$ *only.*

Assume the existence of such an equation, and suppose $A_n \neq 0$. Let $1, 2, \cdots, n, a$ be any $n+1$ points of (B).* Between the equations

$$(3) \qquad \begin{cases} \overline{1,a} = \phi(x_1^1, \cdots, x_n^1; x_1^a, \cdots, x_n^a), \\ \overline{2,a} = \phi(x_1^2, \cdots, x_n^2; x_1^a, \cdots, x_n^a), \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \overline{n,a} = \phi(x_1^n, \cdots, x_n^n; x_1^a, \cdots, x_n^a) \end{cases}$$

can be eliminated just $n-1$ of the coördinates $x_1^a, x_2^a, \cdots, x_n^a$, say $x_1^a, \cdots, x_{n-1}^a$ (Theorem 5). The resulting equation,

$$\xi(\overline{1,a}, \overline{2,a}, \cdots, \overline{n,a}; x_n^a; x_1^1, \cdots, x_n^1; \cdots; x_1^n, \cdots, x_n^n) = 0,$$

algebraic in

$$\overline{1,a}, \overline{2,a}, \cdots, \overline{n,a}, x_n^a, x_1^1, \cdots, x_n^n,$$

becomes an identity upon the substitution of the values of $\overline{1,a}, \overline{2,a}, \cdots, \overline{n,a}$ from (3). Differentiating this identity in turn with respect to $x_1^a, x_2^a, \cdots, x_n^a$ and multiplying successively by $A_1, A_2, \cdots, A_n$, we obtain

$$X(\xi) \equiv A_n \frac{\partial \xi}{\partial x_n^a} \equiv 0,$$

since

$$X(\overline{1,a}) \equiv X(\overline{2,a}) \equiv \cdots \equiv X(\overline{n,a}) \equiv 0.$$

Supposing ξ to be integral and irreducible, we can not have $\partial\xi/\partial x_n^a \equiv 0$. Accordingly, $A_n = 0$, contrary to hypothesis.

§ 3. Construction of the Differential Equations.

Art. 4. Let us take $n+2$ arbitrary points of (B) and designate them

$$1, 2, \cdots, n; a, b.$$

Just one algebraic relation connects the $\frac{1}{2}(n+1)(n+2)$ mutual distances between these points, and this relation is satisfied identically upon the substitu-

* The coördinates $x_1^a, x_2^a, \cdots, x_n^a$ represent the distances between the point a of (B) and n points of (A) which we denoted by $1, 2, \cdots, n$. As these points are not referred to in the future, we shall, for convenience, hereafter reserve the numbers $1, 2, \cdots, n$ for points of (B).

tion of the algebraic functions of the form (1) for the distances (Theorems 1–3). For the purpose of finding these functions we may therefore regard the n coördinates of any point as independent and continuous variables. If we do this, the $\frac{1}{2}(n+1)(n+2)$ functions considered will satisfy only one relation, and as they involve $n(n+2)$ coördinates, we can construct a system of just

$$n(n+2) - [\tfrac{1}{2}(n+1)(n+2) - 1] = \tfrac{1}{2}n(n+1)$$

independent linear partial differential equations whose solutions are the distances considered. Let this system be indicated by M.

For simplicity, we shall write p_i^a instead of $\partial f / \partial x_i^a$, f being any solution of M. Each of the equations of this system is therefore linear and homogeneous in the letters p_i^a, $i = 1, 2, \cdots, n$; $\alpha = 1, 2, \cdots, n, a, b$, and the coefficients are algebraic functions of the variables $x_1^1, \cdots, x_n^n$; $x_1^a, \cdots, x_n^a$; $x_1^b, \cdots, x_n^b$.

The functions $1,a, \cdots, \overline{n,a}$ being independent with respect to $x_1^a, \cdots, x_n^n$ (Theorem 5), we may make a change of variables in M, replacing the variables

$$x_1^a, x_2^a, \cdots, x_n^a; \; x_1^b, x_2^b, \cdots, x_n^b,$$

involved in M, by the variables

$$1,a, \overline{2,a}, \cdots, \overline{n,a}; \; \overline{1,b}, \overline{2,b}, \cdots, \overline{n,b}.$$

The resulting system (M') of differential equations will differ from M merely in being free from the derivatives p_i^a, p_i^b, $i = 1, 2, \cdots, n$, and in having the variables in the coefficients changed. As the system M contains $\frac{1}{2}n(n+1)$ independent equations, the new system M' must contain $\frac{1}{2}n(n+1)$ independent equations, which involve the derivatives p_i^a; $i, \alpha = 1, 2, \cdots, n$, only. Accordingly, we can solve M' for $\frac{1}{2}n(n+1)$ of these derivatives, and therefore also M.

Art. 5. Solving the system M in this manner, let the general solution be indicated by

$$p_j^\beta = [n] + (a) + (b), \tag{4}$$

the derivatives p_j^β being $\frac{1}{2}n(n+1)$ of the derivatives p_i^α; $i, \alpha = 1, 2, \cdots, n$, the remaining of which are included in the symbol $[n]$. The symbols (a) and (b) represent respectively the terms containing the derivatives p_i^a and p_i^b, $i = 1, 2, \cdots, n$, and are derived one from the other by interchanging all the corresponding variables and derivatives of the two points a and b. For, the distances $\overline{\alpha,\beta}$; $\alpha, \beta = 1, 2, \cdots, n, a, b$; $\alpha \neq \beta$, defining the system (4), are unaltered or interchanged by interchanging all the corresponding coördinates of any two of the points $1, 2, \cdots, n, a, b$ (Theorem 4). It follows from this that the system of equations (4) are reproduced identically by interchanging the corresponding coördinates of the two points a and b.

The $\frac{1}{2}n(n+1)$ equations indicated by

$$p_j^\beta = [n] + (a) \tag{5}$$

are independent and are satisfied by the $\frac{1}{2}n(n+1)$ independent distances $\overline{\alpha,\beta}$; $\alpha, \beta = 1, 2, \cdots, n, a$; $\alpha \neq \beta$, which are free from the coördinates $x_1^b, x_2^b \cdots, x_n^b$. As only $\frac{1}{2}n(n+1)$ independent linear partial differential equations can be constructed having for solutions $\frac{1}{2}n(n+1)$ independent functions in $n(n+1)$ variables

$$x_1^1, \cdots, x_n^1;\ x_1^2, \cdots, x_n^2;\ \cdots;\ x_1^n, \cdots, x_n^n;\ x_1^a, \cdots x_n^a,$$

it is easily proved that the equations (5) must be free from the variables $x_1^b, x_2^b, \cdots, x_n^b$; i. e., the terms indicated by the symbol (a) do not contain the coördinates $x_1^b, \cdots, x_n^b$. Similarly, the terms indicated by the symbol (b) do not contain the coördinates $x_1^a, \cdots, x_n^a$.

The system of equations

$$(a) + (b) = 0 \tag{6}$$

must evidently be satisfied by the distance $\overline{a,b}$, whatever be the values of the coördinates x_i^α; $i, \alpha = 1, 2, \cdots, n$, which do not occur in $\overline{a,b}$, but are, likely, involved in the coefficients of (a) and (b). Giving all possible sets of constant values to these n^2 coördinates, we derive from (6) several such equations, l in number, suppose, not satisfying linear relations with constant coefficients. Let any one of these l equations be indicated by

$$[(a)] + [(b)] = 0.$$

Art. 6. If we now form the system of l equations

$$X_i(f) \equiv [(1)] + [(2)] + \cdots + [(n)] + [(a)] + [(b)] = 0 \quad (i = 1, 2, \cdots, l), \tag{7}$$

we readily see that it is satisfied by all the $\frac{1}{2}(n+1)(n+2)$ distances $\overline{\alpha,\beta}$; $\alpha, \beta = 1, 2, \cdots, n, a, b$; $\alpha \neq \beta$. Let k of them be independent. They may then replace k of the system (4), and the remaining $\frac{1}{2}n(n+1) - k$ equations can be taken so as to be free from the letters

$$p_i^a,\ p_i^b,\ x_i^a,\ x_i^b \qquad (i = 1, 2, \cdots, n),$$

owing to the form of the equations (4). Indeed, the coefficients involved in (a) are readily seen to be of the form $\sum_i A_i' F_i$, where A_i' is a function of $x_1^a, x_2^a, \cdots, x_n^a$ only, and F_i is free from these letters and from $x_1^b, x_2^b, \cdots, x_n^b$; the coefficients in (b) have the same form if a and b be interchanged.

These remaining equations must be satisfied by the distances

$$\overline{\alpha,a} \qquad (\alpha = 1, 2, \cdots, n).$$

Substituting we find, for some value of α, an equation of the form

$$\sum_{i=1}^{n} A_i \frac{\partial(\overline{\alpha, a})}{\partial x_i^a} = 0,$$

where the coefficients A_i do not contain the coördinates $x_1^a, \cdots, x_n^a$ as may be easily proved by the reader from what precedes. All other coördinates, excepting $x_1^a, \cdots, x_n^a$, may be put equal to arbitrary constants. There remains an equation of the form debarred by Theorem 6. Accordingly, $k = \frac{1}{2}n(n+1)$, and the system (4), or M, is replaced completely by the system (7).

Let the first $\frac{1}{2}n(n+1)$ of the equations (7) be independent. They form what is called *a complete system of linear partial differential equations*, as

$$n(n+2) = [\tfrac{1}{2}n(n+1)] + [\tfrac{1}{2}(n+1)(n+2) - 1].$$

Therefore, by the theory of linear partial differential equations, the "commutator" of every pair of the left-hand members of (7), as $X_i(f)$ and $X_j(f)$,

$$X_i(X_j(f)) - X_j(X_i(f)),$$

must be linearly dependent upon the expressions

$$X_i(f) \qquad [i = 1, 2, \cdots, \tfrac{1}{2}n(n+1)],$$

namely,

$$X_i(X_j(f)) - X_j(X_i(f)) \equiv \sum_{s=1}^{k} \alpha_s X_s(f) \tag{8}$$

$$[i, j = 1, 2, \cdots, k;\ i \neq j,\ k = \tfrac{1}{2}n(n+1)].$$

These commutators are immediately seen to have the same form as the expressions $X_i(f)$, i. e., they have the form

$$[(1)] + [(2)] + \cdots + [(n)] + [(a)] + [(b)],$$

in which the symbol $[(\alpha)]$ represents a linear and homogeneous expression in the derivatives p_i^a, $i = 1, 2, \cdots, n$, with coefficients which are functions of x_i^a, $i = 1, 2, \cdots, n$, only. Moreover, the expressions represented by two symbols $[(\alpha)]$ and $[(\beta)]$ are obtained one from the other by interchanging all the corresponding coördinates of the points α and β.

We can now prove that the conditions (8) can be satisfied only if the coefficients α_s are constants. For, if not, the integer $v > n$ can be taken great enough that the $\frac{1}{2}n(n+1) + 1$ equations of the form

$$X_i(f) \equiv [(1)] + [(2)] + \cdots + [(v)] = 0 \quad (i = 1, 2, \cdots, k), \tag{9}$$

obtained from the k first of the system (7) and one of the commutators (8) are independent, as the reader may readily prove. The distances

$$(10)\quad \left\{\begin{array}{llllllll} \overline{1,2}, & \overline{1,3}, & \overline{1,4}, \cdots, & \overline{1,t}, \cdots, & \overline{1,n+1}, & \overline{1,n+2}, \cdots, & \overline{1,v}, \\ & \overline{2,3}, & \overline{2,4}, \cdots, & \overline{2,t}, \cdots, & \overline{2,n+1}, & \overline{2,n+2}, \cdots, & \overline{2,v}, \\ & & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ & & \overline{t-1,t}, \cdots, & \overline{t-1,n+1}, & \overline{t-1,n+2}, & \cdots, & \overline{t-1,v}, \\ & & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ & & & \overline{n,n+1}, & \overline{n,n+2}, & \cdots, & \overline{n,v}, \end{array}\right.$$

are solutions of this system and are all independent. For, if they were not, the distances $\overline{1,w}, \overline{2,w}, \cdots, \overline{n,w}$ would satisfy a relation free from the variables $x_1^w, x_2^w, \cdots, x_n^w$, contrary to Theorem 5, w being the highest number occurring in the distances satisfying a tentative relation. (If $w < n+1$, omit $\overline{w,w}$.) The number of solutions (10) being $nv - \frac{1}{2}n(n+1)$, we would have a contradiction, as $\frac{1}{2}n(n+1)+1$ independent equations in nv variables have only $nv - \frac{1}{2}n(n+1) - 1$ independent solutions.

Art. 7. Consider now the $\frac{1}{2}n(n+1)$ expressions indicated by the symbol $[(1)]$ of the first $\frac{1}{2}n(n+1)$ equations of (7). Omitting the index 1 from the variables and derivatives, they are of the form

$$(11)\qquad Y(f) \equiv A_1 p_1 + A_2 p_2 + \cdots + A_n p_n,$$

the coefficients A_i being algebraic functions of $x_1, x_2, \cdots, x_n$, and the letters p_i being the partial derivatives of f with respect to the same coördinates. We have found that the commutator of any pair of the expressions (11) is equal to a sum of these expressions multiplied by constants. If we denote the $\frac{1}{2}n(n+1)$ expressions (11) by $Y_1(f), Y_2(f), \cdots, Y_k(f)$, where $k = \frac{1}{2}n(n+1)$, we have

$$Y_i\big(Y_j(f)\big) - Y_j\big(Y_i(f)\big) \equiv \sum_{s=1}^{k} c_{ijs}\, Y_s(f) \quad (i, j = 1, 2, \cdots, k;\ i \neq j),$$

the coefficients c_{ijs} being constants.

The expressions $Y_1(f), \cdots, Y_k(f)$ can therefore, according to LIE, be considered independent infinitesimal transformations of a k-parametric group, $k = \frac{1}{2}n(n+1)$. This group must possess the following properties:

1. It must possess one, and only one, invariant for two points, which is algebraic by a proper choice of variables; i. e., if $Y^{(a)}(f)$ is any infinitesimal transformation of the group, written in the variables $x_1^a, x_2^a, \cdots, x_n^a$, then must the k equations

$$(12)\qquad Y^{(1)}(f) + Y^{(2)}(f) = 0$$

possess one solution only, which must be algebraic.

2. The condition of Theorem 6 must be observed; i. e., it must be impossible from the k equations (12) to derive a differential equation written only in the variables and derivatives with the index 1. In particular, no two independent infinitesimal transformations can differ only by a factor, for from two such we would get an equation of the form excluded by Theorem 6.

The first condition is readily seen to be true by considering the list of solutions (10) of the equations (9). Only one solution can be obtained from that list containing no other variables than $x_1^1, x_2^1, \cdots, x_n^1; x_1^2, x_2^2, \cdots, x_n^2$.

§ 4. Coincidence of the Groups Defined in Art. 7 with a Class of Groups Defined by Lie.

Art. 8. In his *Untersuchungen über die Grundlagen der Geometrie*,* Lie has defined a class of groups in the following manner: Two points have one, and only one, invariant; s points, $s > 2$, have no invariants independent of the two-point invariants. Such groups have been determined by Lie for S_1, S_2 and S_3,† and by Dr. G. Kowalewski for S_4.‡

We proceed to show that our groups as defined by the conditions 1 and 2 of Art. 7, and the groups defined by Lie are identical, if the algebraic character of the distances (invariants) be disregarded. The groups have the same number of parameters when they are written in the same number of variables n, namely $\frac{1}{2}n(n+1)$. Let

$$X_i(f) \equiv [(1)] + [(2)] + \cdots + [(s)] \quad [i = 1, 2, \cdots, \tfrac{1}{2}n(n+1)] \tag{13}$$

represent the differential equations obtained from our groups or from Lie's groups and corresponding to the $\frac{1}{2}n(n+1)$ independent equations of the system (7), s being taken large enough that they are all independent of each other, and so that the invariants of this system contain at least one of the invariants—if such exist—that do not depend on the two-point invariants. We may take $s > n$. Consider the $ns - \frac{1}{2}n(n+1)$ invariants (distances) (10) (v replaced by s).

They are obviously independent of each other for the groups defined by Lie, and in such a case are the invariants $\overline{1,s}, \overline{2,s}, \cdots, \overline{n,s}$ independent of each other with regard to the variables $x_1^s, x_2^s, \cdots, x_n^s$. Now, this property leads to our condition 2, Art. 7. For, the existence of a differential equation of the form

$$A_1\frac{\partial f}{\partial x_1^s} + A_2\frac{\partial f}{\partial x_2^s} + \cdots + A_n\frac{\partial f}{\partial x_n^s} = 0, \tag{14}$$

* *Theorie der Transformationsgruppen*, Lie-Engel, vol. III, Abtheilung V.

† *Theorie der Transformationsgruppen*, vol. III, p. 6 and pp. 399–437.

‡ *Ueber eine Kategorie von Transformationsgruppen einer vierdimensionalen Mannigfaltigkeiten*, Leipziger Berichte, vol. 50, pp. 60–111.

(the coefficients A_i being functions of $x_1^s, \cdots, x_n^s$ only), and satisfied by the invariants $\overline{1,s}, \overline{2,s}, \cdots, \overline{n,s}$ would deny the independence of these. LIE's groups therefore satisfy our condition 2, and evidently also the condition 1 if we disregard the algebraic form of the invariants.

It remains for us to show that our groups defined by the conditions 1 and 2 possess no invariants (solutions of the system (13)), independent of the two-point invariants (distances).

Firstly, the distances (10) (v replaced by s) are independent of each other, if our condition 2 is satisfied. If they were not, the distances $\overline{1,s}, \overline{2,s} \cdots, \overline{n,s}$ would satisfy a relation free from the variables $x_1^s, x_2^s, \cdots, x_n^s$. Now, the distances $\overline{1,s}, \overline{2,s}, \cdots, \overline{n-1,s}$ define a differential equation of the form (14) with this exception, that we know only that the coefficients A_i do not now contain the variables $x_1^n, x_2^n, \cdots, x_n^n$. This equation would also be satisfied by the distance $\overline{n,s}$ in the case supposed. It is, however, with respect to this function, an equation of the form debarred by Theorem 6. Such an equation could therefore not possibly be derived from our groups, which furnish all the differential equations satisfied by the distance $\overline{n,s}$. Hence, the distances (10) are independent of each other. (It has been assumed that if the functions (10) are not independent, the relation existing among them contains actually one, at least, of the functions $\overline{1,s}, \overline{2,s}, \cdots, \overline{n,s}$. This need not be so, but the analysis is the same if for s we substitute t, this being the highest number occurring in the distances of (10) satisfying a tentative relation. If $t < n+1$, the symbol $\overline{t,t}$ must, of course, be omitted.)

Secondly, the distances (10) (v replaced by s), being all independent, must form a complete set of solutions to the system (13), as the number of independent solutions can be only $ns - \frac{1}{2}n(n+1)$. This system can therefore not contain a solution independent of the distances, and LIE's definition is satisfied by our groups.

Art. 9. We may now take the two-point invariants of the groups obtained by LIE and KOWALEWSKI for S_1, S_2, S_3, S_4, selecting those that can be put into algebraic form by a proper choice of variables. Calling the coördinates x; x, y; x, y, z; x, y, z, u* for spaces of 1, 2, 3, 4, dimensions respectively, the distance $\overline{1,2}$ must be a function of one or other of the following types:

Space of one dimension.

$$x_1 - x_2; \qquad x_2/x_1.$$

* These letters may not represent coördinates in the sense in which such have been defined, i. e., distances between certain fixed points and the variable point, but are certain functions of these distances. To put the groups considered into the typical forms given by LIE and KOWALEWSKI may necessitate a change of variables.

Space of two dimensions.

$$(y_1-y_2)(x_1-x_2)^c,\ c\neq 0; \qquad x_1y_2-x_2y_1;$$

$$\frac{(x_1-x_2)^2+(y_1-y_2)^2+(x_1y_2-x_2y_1)^2}{(1+x_1x_2+y_1y_2)^2}.$$

Space of three dimensions.

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2;$$

$$\frac{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(x_1y_2-x_2y_1)^2+(x_1z_2-x_2z_1)^2+(y_1z_2-y_2z_1)^2}{(1+x_1x_2+y_1y_2+z_1z_2)^2};$$

$$z_1+z_2-2\frac{y_1-y_2}{x_1-x_2}; \qquad z_1-z_2+x_2y_1-x_1y_2;$$

$$z_1-z_2+\frac{(y_2-y_1)^2}{2(x_2-x_1)}; \qquad z_1z_2(x_1-x_2)(y_1-y_2)^c;* \qquad c\neq 0.$$

Space of four dimensions.

Let $\gamma=u_1u_2$,

$$\alpha=(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(u_1-u_2)^2,$$

$$\beta=(x_1y_2-x_2y_1)^2+(x_1z_2-x_2z_1)^2+(x_1u_2-x_2u_1)^2+(y_1z_2-y_2z_1)^2$$
$$+(y_1u_2-y_2u_1)^2+(z_1u_2-z_2u_1)^2.$$

The different types are then

$$\alpha; \qquad \frac{\alpha+\beta}{(1+x_1x_2+y_1y_2+z_1z_2+u_1u_2)^2}; \qquad \gamma[\alpha-(u_1-u_2)^2];$$

$$\gamma(z_1-z_2+x_2y_1-x_1y_2).$$

If the coördinates be considered real numbers, we should add to this list all real functions obtained from those given by an imaginary change of variables. In this manner we would, for instance, obtain the function

$$(x_1-x_2)^2+(y_1-y_2)^2,$$

for space of two dimensions, from the form $(y_1-y_2)(x_1-x_2)^c$.

From each of the types just given we can obtain the corresponding distance-relations by simply eliminating the coördinates involved in the distances concerned. It may be remarked that M. DE TILLY's "condition of six points," stated in his *Essai de géométrie analytique générale*, mentioned above, will hold for all the distance-relations obtained in this manner for S_3; in fact, for all those corresponding to the two-point invariants given by LIE in

* This type is given by LIE in the form $z_1+z_2-\log(x_1-x_2)^2-c\log(y_1-y_2)^2$.

his *Theorie der Transformationsgruppen*, vol. III, pp. 410. This condition, to which DE TILLY attaches prime importance, is as follows: Let the five-point distance-relation connecting the distances of the points a, b, c, d, e be indicated by $(abcde) = 0$. This must have such a form that from any three of the relations $(12345) = 0$, $(12346) = 0$, $(12356) = 0$, $(12456) = 0$, $(13456) = 0$, $(23456) = 0$ must follow the other three. DE TILLY gives only two distance-relations satisfying this condition, namely those existing for the euclidean and non-euclidean spaces, corresponding to the two first distances given above for space of three dimensions.

Art. 10. In conclusion it may be pointed out that in the case of the spaces we have defined and discussed, it is not certain that a given equation among the coördinates will define a locus in the ordinary sense, there may be no points, or a finite number only, whose coördinates satisfy the given equation.

There is, however, always one locus in any S_n for which such an equation exists. By Axiom III, a S_{n-1} is contained in the S_n considered, and it is clear from Theorem 2 and the arguments following that the n coördinates of the points of this S_{n-1} satisfy an algebraic equation, and only one such. In any S_n there is therefore always a S_1, which we might call a straight line, the n coördinates of whose points satisfy $n - 1$ independent algebraical equations.

STANFORD UNIVERSITY,
July, 1902.

ON SUPEROSCULATING QUADRIC SURFACES*

BY

HEINRICH MASCHKE

In trying to determine a surface of the second order $F(x, y, z) = 0$ having with a given surface Φ, whose coördinates are given as functions of u and v, a contact of the third order, ten equations are obtained, viz., F and all the first, second and third derivatives of F with respect to u and $v = 0$.

It seems as if after the elimination of the nine constants of F one equation for u, v would result, i. e., one curve on Φ would exist in every point of which a contact of the above description takes place. HERMITE, however, has shown that this is not so. † He found that in general *two* equations result which are free from the coefficients of F, determining a finite number of points P on Φ, and that for every such point P there exists not only one surface F but a whole pencil having a contact of the third order with Φ. ‡

Instead of actually deducing the resulting two conditions by elimination from the above named ten equations we can obtain them in the following much simpler way, which at the same time affords a better insight into the nature of the points in question.

Let us consider the two straight lines a_1 and a_2 of F through a contact point P. Each of them has with a normal section of Φ through P a contact of the third order. These lines a_1 and a_2 are the inflexional tangents of Φ at P; they have then *four* points with the surface Φ at P in common. § Thinking of the inflexional tangents as osculating circles of the two respective normal sections of Φ, they appear now as those particular osculating circles having with their normal sections a contact of the *third* order. Of normal sections of this kind there exist in general three through every point of a surface. They are defined by the following equation: ||

* Presented to the Society (Chicago) March 29, 1902, under a somewhat different title. Received for publication September 2, 1902.

† HERMITE, *Cours d'Analyse* (1873), pp. 148, 149.

‡ A general investigation of this interesting fact also for superosculating surfaces of orders higher than 2, has been given by HALPHEN: *Sur le contact des surfaces*, Bulletin de la Société Mathématique de France, III, pp. 28–37.

§ An algebraic discussion of these points on algebraic surfaces has been made by CLEBSCH, Journal für reine und angewandte Mathematik, vol. 63 and SCHUBERT, Mathematische Annalen, vol. 11, p. 347.

|| KNOBLAUCH, *Einleitung in die allgemeine Theorie der krummen Flächen*, p. 94 (8).

$$(1) \qquad Pdu^3 + 3Qdu^2dv + 3Rdu\,dv^2 + Sdv^3 = 0.$$

It now follows that the $dv : du$ belonging to the inflexional tangents must satisfy the above equation, i. e., since the inflexional tangents are defined by the equation

$$(2) \qquad Ldu^2 + 2Mdu\,dv + Ndv^2 = 0,$$

the left side of (2) must be a factor of the left side of (1). The condition that a quadratic form be a factor of a cubic form is given by the identical vanishing of a certain cubic covariant. * This yields for our case the following four equations:

$$(3) \qquad \begin{cases} (4M^2 - LN)P - 6LMQ + 3L^2R = 0, \\ L^2S - 3LNQ + 2MNP = 0, \\ N^2P - 3LNR + 2LMS = 0, \\ (4M^2 - LN)S - 6MNR + 3N^2Q = 0. \end{cases}$$

These equations are in every given case equivalent to *two* only, since two of them can be deduced from the two others (in order to cover however all possible cases all four equations are necessary). *These are then the required two relations* defining in general on Φ a certain finite number of points P.

The case that the two straight lines a_1 and a_2 of F through P coincide, from which immediately also the coincidence of the two inflexional tangents of Φ follows, does not require any modification of the equations (3). In this case we have however

$$(4) \qquad LN - M^2 = 0,$$

and by means of this relation the equations (3) can be transformed into

$$(5) \qquad \begin{cases} NP - 2MQ + LR = 0, \\ NQ - 2MR + LS = 0. \end{cases}$$

The left sides of these equations are equal to

$$(EG - F^2)\frac{\partial K}{\partial u} \quad \text{and} \quad (EG - F^2)\frac{\partial K}{\partial v}$$

respectively, † where K denotes the Gaussian curvature. We have then for our case (4):

$$(6) \qquad \frac{\partial K}{\partial u} = 0, \qquad \frac{\partial K}{\partial v} = 0.$$

* CLEBSCH, *Theorie der binären algebraischen Formen* (1872), p. 94.
† KNOBLAUCH, *loc. cit.*, p. 97 (5).

One might ask what surfaces Φ are such that in every point a contact of the third order with a F_2 is possible. Then the equations (3) must hold for every point of the surface.

Let us, in order to answer this question, in the first place suppose the surface Φ to be a developable surface. Then (4) holds, i. e., $K = 0$. But since now $K = 0$ for every point of the surface we have

$$\frac{\partial K}{\partial u} = 0, \qquad \frac{\partial K}{\partial v} = 0,$$

i. e., (6) from which conversely the equations (3) follow.

In the second place, when Φ is not a developable surface we can take the asymptotic curves on it as parameter lines. We assume then

$$L = 0, \qquad N = 0, \qquad M \neq 0.$$

The equations (3) are then reduced to

$$P = 0, \qquad S = 0,$$

from which immediately

(7) $$J_2 = 0, \qquad J_1'' = 0$$

follows.* Now in general the geodesic curvature of the parameter-lines on a surface is given by the formulas †

$$g_u = \sqrt{\frac{EG - F^2}{G^3}} \cdot J_1'', \qquad g_v = \sqrt{\frac{EG - F^2}{E^3}} \cdot J_2.$$

We see then from (7) that the asymptotic curves on Φ have everywhere the geodesic curvature zero. But for asymptotic curves the geodesic curvature is identical with the ordinary (first) curvature. The asymptotic curves are then straight lines and the surfaces Φ are surfaces of the second order.

The developable surfaces and the surfaces of the second order are then the only ones for every point of which a surface of the second order exists having a contact of the third order.

THE UNIVERSITY OF CHICAGO,
September 1, 1902.

* KNOBLAUCH, *loc. cit.*, p. 96 (14).
† KNOBLAUCH, *loc. cit.*, p. 248 (15).

A DEFINITION OF ABSTRACT GROUPS*

BY

ELIAKIM HASTINGS MOORE

INTRODUCTION.

Dr. E. V. HUNTINGTON has recently given two different definitions of abstract groups by sets of respectively three† and four independent postulates; these definitions were perhaps suggested by those given by WEBER and by BURNSIDE.

Some years ago Professor J. PIERPONT and I independently hit on a type of definition very desirable from the grouptheoretic standpoint.

In § 1 I formulate a definition of this type by means of five independent postulates; the independence is proved in § 4. In § 3 the definition is related to other definitions. In this connection I am led to a slightly modified definition (p. 489) by means of six independent postulates, which, even from the standpoint of abstract logic, seems to me simpler than either of those of Dr. HUNTINGTON.

§ 1. THE DEFINITION.

We have for consideration a set‡ of elements and a multiplication-table or rule of combination whereby to every two elements a, b taken in the definite order a, b there corresponds a definite so-called product, in notation $a \circ b$, or, when without confusion, more simply, ab; this product may or may not be an element of the set. This set of elements, as related by the multiplication-table, constitutes a group in case the following postulates are fulfilled, viz.,

(1) For every two elements a, b the product ab is an element of the set.

(2) The associative law is fulfilled, that is, $(ab)c = a(bc)$, for every three elements a, b, c such that the products $ab, bc, (ab)c$ and $a(bc)$ are elements of the set. §

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* Presented to the Society April 26, 1902. Received for publication September 17, 1902.

† In fact, four, for the third postulate consists of two parts, each of which is used in the development of the theory. The question of the independence of the four postulates arises; for finite groups either part is redundant. – Cf. § 3.

‡ The set is supposed to contain at least one element.

§ For use in § 3, we note here the stronger statement:

(2′) For every three elements a, b, c such that the products ab, bc and either $(ab)c$ or $a(bc)$ are elements of the set the associative law is fulfilled.

This is a double statement; we denote its two parts by $(2'_1), (2'_2)$.

(3_l) There exists a left-hand identity element, that is, an element i_l such that, for every element a, $i_l a = a$.

(3_r) There exists a right-hand identity element, that is, an element i_r such that, for every element a, $a i_r = a$.

(4_l) If there exists a right-hand identity element, then for some such element i_r it is true that for every element a there exists a left-hand reciprocal element, that is, an element a'_l such that $a'_l a = i_r$.

To these five postulates of definition of abstract groups in general the addition of a sixth postulate (5_α) or (5_β) serves to discriminate between the groups of the various finite orders and those of infinite order: $N = n$, $N = \infty$; viz.:

(5_α) The number of elements is a certain finite integer n.

or

(5_β) The set contains an infinitude of elements.

Remark. The statement (4_r) differing from (4_l) by the interchange of the terms *right* and *left* and introducing right-hand reciprocal elements a'_r is a theorem; cf. theorems (6, 7) of § 2. In the definition the statement (4_r) might of course enter as a postulate in the place of (4_l).

§ 2. Auxiliary theorems.

In § 4 the independence of the postulates of the set ($1, 2, 3_l, 3_r, 4_l, 5_\alpha$) for $n \geqq 3$ and likewise the independence of those of the set ($1, 2, 3_l, 3_r, 4_l, 5_\beta$) will be proved.

In § 3 the definition of § 1 will be related to other definitions; for this purpose the following theorems are needed.

(6) Any left-hand identity element i_l is equal to any right-hand identity element i_r. Hence, there is a single identity element i_l, and likewise there is a single identity element i_r, and these are the same element, say i, the identity element of the group.

(7) For every element a a left-hand reciprocal element a'_l is likewise a right-hand reciprocal element, that is, in view of theorem (6), $a a'_l = i$.

(8) For every element a any left-hand reciprocal element a'_l is equal to any right-hand reciprocal element a'_r. Hence, there is a single left-hand reciprocal element a'_l, and likewise there is a single right-hand reciprocal element a'_r, and these are the same element, say a^{-1}, the reciprocal of the element a in the group.

These statements follow from the equations:

$$(6°) \qquad i_l = i_l i_r = i_r;$$

$$(7°) \quad a a'_l = i(a a'_l) = (a''_l a'_l)(a a'_l) = \{a''_l (a'_l a)\} a'_l = (a''_l i) a'_l = a''_l a'_l = i;$$

$$(8°) \qquad a'_l = a'_l i = a'_l (a a'_r) = (a'_l a) a'_r = i a'_r = a'_r,$$

where in (7°) and (8°) the identity element i of theorem (6) enters, and where further in (7°) a'' denotes a left-hand reciprocal element of the element a_i'.

(9′, 9″). For every two elements a, b there exists one (theorem 9′) and only one (theorem 9″) element x such that $ax = b$.

(10′, 10″). For every two elements a, b there exists one and only one element y such that $ya = b$.

(11′, 11″). For every two elements a, b there exists an element z and there exists an element w such that $(az)b = b, \quad b(wa) = b$.

These are precisely the elements $x = a^{-1}b$, $y = ba^{-1}$, $z = a^{-1}$, $w = a^{-1}$.

§ 3. Comparison with other definitions.

We consider the definitions of Weber * and of Huntington †. These definitions postulate for the set of elements with a multiplication-table the following respective sets of statements:

(W_1): $(1, 2, 9', 9'', 10', 10'')$; $\quad (W_2)_{N=n}$: $(1, 2, 9'', 10'', 5_\alpha)$;

(H_1): $(2', 9', 10')$; $\quad (H_2)$: $(1, 2, 11', 11'')$.

We denote by $(H_1)_{N=n}$ or $(H_1)_{N=\infty}$ the set of postulates (H_1) with the addition of (5_α) or (5_β), and so in general.

Weber remarks that in $(W_1)_{N=n}$ the postulates $(9', 10')$ are redundant, and thus obtains his definition $(W_2)_{N=n}$ of finite groups.

Huntington exhibits the equivalence ‡ of the various definitions $(W_1; H_1; H_2)$ of groups in general, $(W_2; H_1; H_2)_{N=n}$ of finite groups, and $(H_1; H_2)_{N=\infty}$ of infinite groups; and he proves the independence of the postulates of every one § of these definitions, with the exception of (W_1) in which $(9'', 10'')$ are redundant, as one sees by use of (H_1). Further he remarks that $(W_2)_{N=\infty}$ is not a definition of infinite groups, the set of positive integers with $a \circ b = a + b$ being a non-group example.

I call attention to the following definitions:

$$(W_1')\text{: } (1, 2, 9', 10'); \quad (W_1')_{N=n}; \quad (W_1')_{N=\infty},$$

of the three types of groups, and raise the question of the independence of the postulates of these definitions. The definition (W_1') is from the grouptheoretic standpoint a more convenient modification of Weber's definition (W_1) than is

* *Lehrbuch der Algebra*, vol. 2 (1899), pp. 3, 4.

† Bulletin, ser. 2, vol. 8 (April and June, 1902), pp. 296–300, 388–391.

‡ As to the theory of (H_1), it should be remarked that the statements 6 and 7 of *l. c.*, pp. 297, 298 (as proved and as used) involve the (unstated) hypothesis that $a \circ b$ is of the set or assemblage. Of course the stronger statements are likewise true, for in 10, p. 299, it is proved that every product $a \circ b$ is of the set.

§ As to $(H_1)_{N=\infty}$, the proof for (H_1) is readily made effective.

(H_1). In my lectures of the autumn of 1900 I used this definition (W_1') and the definition *

$$(M'): \quad (1, 2, 3_l, 3_r, 4_l, 4_r),$$

proving as in § 2 that the statements of (W_1') follow from those of (M'), and further that those of (M') follow from those of (W_1') by the use of the theorems:†

(ρ). If an element b is a right-hand identity element for a certain element a_0 ($a_0 b = a_0$), then it is for every element a ($ab = a$).

(λ). Similarly: If $ba_0 = a_0$, then $ba = a$.

The redundance of (4_r) in (M') and the independence of the remaining postulates, those of the new definition (of § 1),

$$(M): \quad (1, 2, 3_l, 3_r, 4_l),$$

and likewise the independence of the postulates of the corresponding definitions $(M)_{N=n}$, $(M)_{N=\infty}$ were discovered in April, 1902.

The four definitions of groups in general are most closely related in the following order:

$$(M), \qquad (W_1'), \qquad (H_1), \qquad (H_2);$$

the redundance of $(9'', 10'')$ in (W_1) is a consequence of the fact that the statements of (M) follow from those of (W_1').

From the standpoint of abstract logic the canons of relative simplicity of equivalent definitions by sets of postulates are not well established. Perhaps the only established canon is this, that a definition is simplified by the omission of a group of postulates logically deducible from the remaining postulates. One is tempted to add this, that every postulate of a desirably simple definition shall be a simple statement, that is, a single and not a multiple statement. The difficulty here would arise in the precise formulation of the terms of this second canon, especially in view of the fact that the same statement may be made in various forms. At least, a definition is simplified by the substitution, for a

* The earlier definitions of PIERPONT and of myself (already referred to) introduce an identity element i and for every element a a reciprocal element a^{-1} by the postulates (3, 4) that they exist and satisfy respectively for every element a the two double equations:

$$(3^\circ) \quad ia = ai = a, \qquad (4^\circ) \quad a^{-1}a = aa^{-1} = i.$$

PIERPONT's definition:

$$(P)_{N=n}: \quad (1, 2, 3, 3^*, 4, 5_a),$$

involved further the postulate (3*) of the uniqueness of the identity element and it is given for finite groups (Annals of Mathematics, ser. 2, vol. 2, p. 47, Oct. 1900, in the report of the lectures of the Buffalo Colloquium of September, 1896, on *Galois's theory of algebraic equations*).

The definition:

$$(1, 2, 3, 4)$$

of groups in general I used in lectures in January, 1897.

† The lemmas 4, 5 (p. 297) of HUNTINGTON's theory of (H_1).

postulate consisting of an aggregate of independent statements, of those statements as distinct postulates. Further one may add as a third canon this, that of two definitions one with the smaller number of postulates is the simpler. As to this canon the case now in question seems to show that the definition with the larger number of independent postulates may reveal more immediately the fundamental properties of the object of definition. It is, of course, evident that the task of proving the independence of the postulates presumably increases with the number of postulates.

Let us compare the four definitions (W_1'), (H_1), (H_2), (M) of groups in general on the basis of these three canons.

For (W_1') the independence of the postulates is an open question; for the other definitions it has been proved.

The postulate $(2')$ of (H_1) consists obviously of the two parts $(2_1')$, $(2_2')$, and thus the question of independence of the four postulates of the new definition *

$$(H_1'): \quad (2_1', 2_2', 9', 10')$$

arises. No other postulate of the four definitions is similarly multiple.

However, for instance, postulate (1) of (W_1'), (H_2) and (M) breaks into the aggregate of statements $(1)_{a,b}$, a, b denoting any two elements of the set, viz.,

$(1)_{a,b}$. The product ab of the two elements a, b is an element of the set.

Similarly, all the postulates of the four definitions (W_1'), (H_1'), (H_2), (M), with the exception of the postulates $(3_l, 3_r, 4_l)$ of (M), are at once decomposable into aggregates of constituents.

That the postulates $(3_l, 3_r, 4_l)$ are not thus decomposable may properly be considered an indication of their greater intrinsic complexity. Accordingly I replace (M) by the new definition:

$$(M''): \quad (1, 2, 3'', 3_l'', 3_r'', 4_l''),$$

by means of six independent postulates. Here the new postulates are these:

$(3'')$. There exists at least one idempotent element, that is, an element i identical with its square, $ii = i$.

$(3_l'')$. Every† idempotent element is a left-hand identity element; that is, for every element a and every† idempotent element i, $ia = a$.

* One readily sees that, if for three elements a, b, c the hypotheses of $(2_1')$ and $(2_1')$ itself are satisfied, $(2_2')$ is likewise satisfied for those elements, and, accordingly that the complete postulate $(2_2')$ is redundant in $(H_1')_{N=n}$, for, as HUNTINGTON has remarked, from $(9', 10', 5^a)$ follows (1) at once.

† The statements $(3_l'', 3_r'', 4_l'')$ have as hypotheses the existence of at least one idempotent element. Accordingly, in any case in which $(3'')$ is not satisfied, $(3_l'', 3_r'', 4_l'')$ are satisfied in that their hypotheses are not fulfilled, or, as we may say, are satisfied vacuously. This type of validity of postulates is of considerable importance.

$(3''_r)$. Every * idempotent element is a right-hand identity element.

$(4''_l)$. For every element a with respect to every * idempotent element i there exists a left-hand reciprocal element, that is, an element $a'^{(i)}_l$ such that $a'^{(i)}_l a = i$.

It is easy to see that the sets of statements $(3_l, 3_r)$, $(3'', 3''_l, 3''_r)$ are equivalent, and that, if either set is valid, there is but one idempotent element, the identity element of (6), and hence that with respect to these equivalent sets of statements the statements (4_l) and $(4''_l)$ are equivalent.—Accordingly, the proof to be given in § 4 of the independence of the postulates of (M), $(M)_{N=n}$, $(M)_{N=\infty}$ respectively applies at once to prove the independence † of the postulates, except $(3'', 3''_l, 3''_r)$, of (M''), $(M'')_{N=n}$, $(M'')_{N=\infty}$ respectively. One proves the independence of $(3''_l)$ and that of $(3''_r)$ for $N = n \geqq 2$ or $N = \infty$ by the examples given in § 4 for (3_l) and (3_r) respectively.

The postulate $(3'')$ is independent in (M'') and in $(M'')_{N=\infty}$, as one sees by the set of elements a_h with positive integral indices h with the multiplication-table $a_{h_1} a_{h_2} = a_{h_1+h_2}$. In the definition $(M'')_{N=n}$ of finite groups, however, the postulate $(3'')$ is redundant. For in any finite set of elements with multiplication-table satisfying (1, 2) there exists a closed cycle of (one or more) elements, each of which is the square of the preceding element in the cycle; these elements are in multiplication commutative, being all of them powers of any one; their product is an idempotent element, for the square of that product is the product of the squares of the elements, that is, it is that product itself.

To recur to the comparison of the definitions (W'_1), (H'_1), (H_2), (M'').—The new postulates $(3''_l, 3''_r, 4''_l)$ are decomposable in the manner desired. For the new definitions arising by this decomposition of the postulates the question of independence arises. In none of these new definitions are all the postulates independent. For, in the first place, upon the statements (1), (2) and respectively

$$(9')_{a,b}, \qquad (10')_{a,b}, \qquad (11')_{a,b}, \qquad (11'')_{a,b}$$

depend respectively the statements

$$(9')_{ca,cb}, \qquad (10')_{ac,bc}, \qquad (11')_{a,bc}, \qquad (11'')_{a,cb}$$

for every element c. And, as already noted, for every postulate $(2'_1)_{a,b,c}$ non-vacuously verified, the corresponding postulate $(2'_2)_{a,b,c}$ is redundant. And, further, the postulate $(4''_l)^{(i)}_{ab}$ depends upon the postulates $(1, 2, 3'', 3''_l, 3''_r)$ and $(4''_l)^{(i)}_a, (4''_l)^{(i)}_b$; for, in the notations of $(4''_l)$, since $(b'^{(i)}_l a'^{(i)}_l)(ab) = i$, the element $b'^{(i)}_l a'^{(i)}_l$ is an element $(ab)^{(i)}_l$ satisfying the condition of postulate $(4''_l)^{(i)}_{ab}$.

* Cf. footnote † on the preceding page.

† For $N = n > 2$ or $N = \infty$. — As to the cases $N = n = 1, 2$ one readily finds that for $n = 1$ all the postulates depend upon (1) and $(5_a)_{a=1}$ which are independent, and that for $n = 2$ the postulates $(3'')$, $(3''_l)$, $(3''_r)$, $(4''_l)$ and $(5_a)_{a=2}$ are independent, while (1), (2) depend upon them.

Finally, it seems worth while in a fairly definite way to apply the third canon of comparison to the four definitions. As a unit-operation of the determination that a given set of N elements with a given multiplication-table constitutes a group we consider the reading from the table of a single entry ab, with necessary checking.

The verification of postulate (1) requires N^2 of these unit-operations, while (2), $(2'_1)$ require each $4N^3$ operations.* Similarly (9'), (10'), (11'), (11'') require each at least N^2 and at most N^3 operations, while (3'') requires at least 1 and at most N operations, and then † $(3''_l)$, $(3''_r)$ require each $N-1$ operations, and $(4''_l)$ at least $N-1$ and at most N^2-N operations. Thus the verification of the group-property of the given set of N elements according to the various definitions requires

	at least	at most
$(W'_1)=(1, 2, 9', 10')$:	$4N^3+2N^2$,	$6N^3$;
$(H'_1)=(2'_1, 2'_2, 9', 10')$:	$4N^3+2N^2$,	$6N^3$;
$(H_2)=(1, 2, 11', 11'')$:	$4N^3+2N^2$,	$6N^3$;
$(M'')=(1, 2, 3'', 3''_l, 3''_r, 4''_l)$:	$4N^3+3N-2$,	$4N^3+N^2+2N-2$

operations. Here it is understood that N, N^2, N^3, and the symbols of addition and multiplication have a rather definite meaning, even if $N=\infty$, in connection with the notion of reading from the multiplication-table.

Thus, as stated in the introduction, the definition (M'') seems to me to be an advantageous one both from the grouptheoretic and the logical standpoints.

It remains to prove the independence of the postulates of (M).

§ 4. The independence of the postulates.‡

The postulates of the set $(1, 2, 3_l, 3_r, 4_l, 5_\alpha)$ or $(1, 2, 3_l, 3_r, 4_l, 5_\beta)$ are proved independent by the exhibition of six sets of elements with multiplication-tables satisfying all the postulates but one.

As to (5_α) or (5_β). We consider any group containing an infinitude or a finite number of elements.

As to (1).—We consider the elements $a_{g\,h}$ with the multiplication-table

$$a_{g_1\,h_1}a_{g_2\,h_2}=a_{g_1+g_2\;h_1+h_2}.$$

* On finding (2) or $(2'_1)$ verified for every triad a, b, c of elements, one sees that (1) or $(2'_2)$ is verified.

† Since there is only one idempotent element, if for one idempotent element the conditions $(3''_l, 3''_r)$ are satisfied.

‡ For $N=n>2$ or $N=\infty$. As to the cases $N=n=1, 2$ one readily finds that for $n=1$ all the postulates depend upon (1) and $(5_\alpha)_{n=1}$ which are independent, and that for $n=2$ the postulates (1), (2) depend upon (3_l), (3_r), (4_l) and $(5_\alpha)_{n=2}$ which are (as proved in the text) independent.

Here the first index is an integer taken modulo 2 and the second index is an integer taken either modulo $2m$ or without modular condition.

The set of all such elements with this multiplication-table constitutes a group of order $N = 2.2m$ or $N = \infty$.

The non-group sets of order $N = n$ or $N = \infty$ satisfying all the postulates except (1) are obtained from this set of elements with properly chosen m by the omission of certain elements.

In case $N = n \geqq 3$, we take $m \geqq 2$ so that $n = 2m - 1$ or $2m$, and omit from the $2.2m$ elements a_{gh} these elements: a_{0m} and the $2m$ or $2m - 1$ elements $a_{1h}(h = 0, 1, 2, \cdots, 2m - 1$ or $h = 1, 2, \cdots, 2m - 1)$. The product $a_{01}a_{0m-1} = a_{0m}$ is not in this set.

In case $N = \infty$, we omit the elements $a_{1h}(h = \pm 1, \pm 2, \pm 3, \cdots)$. The product $a_{01}a_{10} = a_{11}$ is not in this set.

As to (2).—The non-group example is the set of $N = n \geqq 3$ or $N = \infty$ elements i and a_h where $h = 1, 2, \cdots, n - 1$ or $h = 1, 2, 3, \cdots$ with the multiplication-table

$$ii = i, \qquad ia_h = a_h i = a_h, \qquad a_h a_h = i, \qquad a_{h_1}a_{h_2} = a_{h_1} \quad (h_1 \neq h_2).$$

Here

$$a_1(a_1a_2) = a_1a_1 = i, \quad \text{while} \quad (a_1a_1)a_2 = ia_2 = a_2.$$

As to (3_l).—The non-group example is the set of $N = n \geqq 2$ or $N = \infty$ elements a_h where $h = 1, 2, 3, \cdots, n$ or $h = 1, 2, 3, \cdots$, with the multiplication-table

$$a_{h_1}a_{h_2} = a_{h_1}.$$

Here there is no left-hand identity element.

As to (3_r).—The example for $N = n \geqq 2$ or $N = \infty$ is the example for (3_l) with the modified multiplication-table

$$a_{h_1}a_{h_2} = a_{h_2}.$$

As to (4_l).—The non-group example for $N = n \geqq 2$ or $N = \infty$ is the set for (2) with the multiplication-table

$$ii = i, \qquad ia_h = a_h i = a_h, \qquad a_{h_1}a_{h_2} = a_1.$$

Here the only right-hand identity element is i, and there is no element x such that $xa_1 = i$.

OCEAN VIEW, VA.,

September 17, 1902.

ALGEBRAIC TRANSFORMATIONS OF A COMPLEX VARIABLE REALIZED BY LINKAGES*

BY

ARNOLD EMCH

1. In the Comptes Rendus† of 1895, Professor KOENIGS has proved the following very interesting theorem:

"*Let $M_1, M_2, \cdots, M_n$ be n points connected by an algebraic relation. This algebraic relation can be realized by a linkage.*"

If $x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; \cdots; x_n, y_n, z_n$ are the rectangular coördinates of the n points, the algebraic relation considered may be written in the form

$$f(x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_n, y_n, z_n) = 0. \tag{1}$$

where f is a polynomial in $x_1, y_1, z_1, \cdots, x_n, y_n, z_n$ with real coefficients.

KOENIGS also shows that *the theorem holds when the n points are connected by any number of algebraic relations, provided this number does not make a rigid system of the n points;* the linkage in question is obtained by uniting into one linkage the linkages corresponding respectively to the various relations, in such a way that the points of similar designation in these linkages are spatially identical.

The purpose of this paper is to show how this theorem may be specialized for complex variables.

2. In the first place it is easily seen from KOENIGS's proof that the theorem also holds for the plane, i. e.,

Any number of algebraic relations between n points in a plane can be realized by a plane linkage.

Also in this case the number of relations must be such that the system is movable.

To have a definite idea about the character of the plane linkages to be considered I set down KOENIGS's definition:

* Presented to the Society (Chicago) January 2, 1902. Received for publication March 13, 1902.

† No. 16, p. 861 and no. 18, p. 981. See also

KOENIGS: *Leçons de Cinématique* (1897), pp. 271–308, and in particular p. 302.

KLEIBER: *Beitrag zur kinematischen Theorie der Gelenkmechanismen*, Zeitschrift für Mathematik und Physik, vol. 36 (1891), pp. 296, 328; vol. 41 (1896), pp. 233, 281.

A plane linkage (*système articulé plan, Gelenkwerk*) *is a combination of plates or plane figures subject to remain in one and the same plane, among which a certain number are connected to each other by hinges or pivots perpendicular to the common plane.*

In this definition it is assumed that the links move by each other without interference, which means that the links, considered as material, lie in a series of close parallel planes.

Every linkage is constructed in such a manner that one of its pivots is fixed and represents the origin O, while others represent the algebraically related variables. The points of the linkages shall always be designated by the same letters as the corresponding variables.

Two or more linkages each involving two variables may be combined in the following manner: Suppose L, L_1, $L_2 \cdots L_n$ are linkages realizing the transformations

$$u = f(u_1),\ u_1 = f_1(u_2),\ \cdots,\ u_{n-1} = f_{n-1}(u_n),\ u_n = f_n(z).$$

Let the origins of all these linkages coincide; attach the pivot u_n of L_n to the pivot u_n of L_{n-1}; attach the pivot u_{n-1} of L_{n-1} to u_{n-1} of L_{n-2}, and so forth; finally the pivot u_1 of L_1 to u_1 of L. Then, the point u of L evidently realizes the compound transformation

$$u = f\left\{f_1\left[f_2 \cdots f_{n-1}\left(f_n(z)\right)\right]\right\} = F(z). \tag{2}$$

Linkages involving more than two variables may be similarly combined.

The range of effectiveness of a linkage is, of course, limited to a certain finite portion of the plane. This range, although in some cases small, always exists.

3. The proofs of KOENIGS's theorems are based upon the consideration of real quantities. Since, however, an algebraic relation amongst n complex variables $z_1 = x_1 + iy_1, \cdots, z_n = x_n + iy_n$ is equivalent to a pair of real algebraic relations amongst the n coplanar points $(x_1, y_1), \cdots, (x_n, y_n)$, we have the theorem:

Any number of algebraic relations between n complex variables may be realized by a plane linkage.

4. We consider more closely the special case of one algebraic relation

$$f(u, z) = 0 \tag{3}$$

between the complex variables u, z. It is clear that a linkage for the relation

$$w = f(u, z) \tag{4}$$

becomes by fixing w at the origin a linkage for the relation (3), and that a link-

age for the relation (4) results from a suitable combination of a number of linkages for the relations

(5) $$z = z_1 + z_2, \qquad z = z_1 z_2.$$

Hence linkages for these fundamental relations (5) have especial interest. Such linkages I proceed to exhibit, the first devised by myself and the second * devised by KLEIBER (loc. cit.).

5. *Addition, Subtraction, Translation.*

Two complex numbers z_1 and z_2 may be added or subtracted by the linkage of Fig. 1, in which all links are equal. Only the point o is fixed. A glance

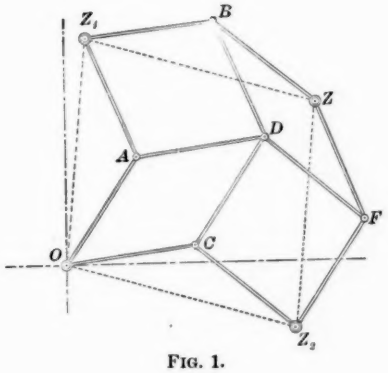

FIG. 1.

at the figure shows that $oz_1 \parallel z_2 z$ and $oz_2 \parallel z_1 z$, no matter how the linkage may be deformed; hence $z = z_1 + z_2$. From the figure it is seen without difficulty that the common range of z_1 and z_2 is a circle having o as a center and twice the length of one link as a radius and that the range of z is a concentric circle with four times the length of one link as a radius. If z_1 and z are given, the point z_2 effects the subtraction $z_2 = z - z_1$. If one of the points, say $z_2 = a$ is constant, then the linkage produces the translation $z = a + z_1$.

6. *Multiplication, Division, Rotation* (KLEIBER).

In order to have a clear understanding of KLEIBER's linkage consider first the "chain" of Fig. 2,* where the shaded triangles are all similar and the remainder of the figure consists of three parallelograms whose connection with the triangles is evident from the figure. It can easily be proved that the triangles

* Independently I had devised linkages for $z_1 = az$, $z_1 = z^n$, where a is any complex constant and n is any rational number.

* See KLEIBER, loc. cit., and also W. DYCK's *Katalog mathematischer Modelle*, etc., pp. 318–328.

$A_0D_1A_2$ and $A_1D_2A_3$ and consequently also A_0CA_3 are similar to the shaded triangles. This property holds no matter how the linkage may be distorted.

Combining now two of these linkages with the points A_0, A_1, A_2, A_3 pivoted together, and with the links A_0A_1, A_1A_2, A_2A_3 left out, Fig. 3, and connecting

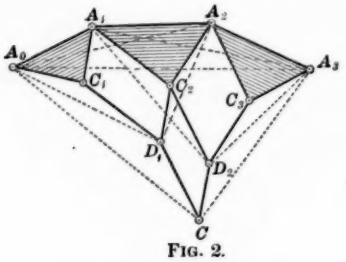

FIG. 2.

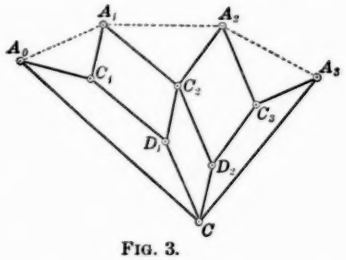

FIG. 3.

A_0C, CA_3, A_3D, DA_0 by links, *so that the quadrilateral* A_0CA_3D *is similar to the quadrilateral* $A_0C_1A_1D_1$, and as a consequence similar to $A_1C_2A_2D_2$ and $A_2C_3A_3D_3$, KLEIBER's linkage, Fig. 4, is obtained. These quadrilaterals remain

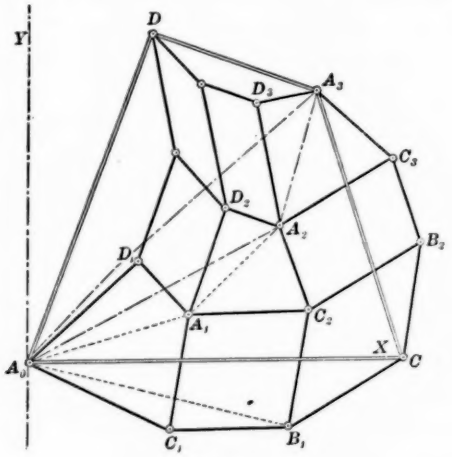

FIG. 4.

similar no matter how the linkage may be distorted. To prove this consider the linkage of Fig. 3, which is the same as that of Fig. 2 with the links A_0A_1, A_1A_2, A_2A_3 left out, but with two new links A_0C and A_3C for which

$$A_0C : A_3C = A_0C_1 : A_1C_1.$$

This linkage when distorted contains the similar triangles of (2) as a special case.

In (3) A_0, A_1, A_2 may be taken arbitrarily, but then A_3 is determined. We make the ratios $A_0A_1 : A_1A_2$, $A_0C_1 : A_1C_2$ equal; then

$$\Delta A_2A_3C \sim \Delta A_0A_1C_1 \sim \Delta A_1A_2C_2 \sim \Delta A_0A_3C.$$

Now in KLEIBER's linkage two linkages (4) X and Y are compounded at the corresponding points A_0, A_1, A_2, A_3. If X and Y have the three points A_0, A_1, A_2 (arbitrarily given) in common, then the A_3's will not coincide. But if A_0, A_1, A_2 are chosen so that $\Delta A_0C_1A \sim \Delta A_1C_2A_2$, then the A_3's of X and Y coincide. This follows from the similitude of the different quadrilaterals and triangles determined by the points A_0, A_1 and A_2. In other words, the compound linkage is movable only when the corresponding triangles and quadrilaterals previously mentioned are similar.

From Fig. 4 it is not difficult to prove the similitude of a number of triangles. Thus, $\Delta A_0A_2A_3 \sim \Delta A_0B_1C$. If A_0C is kept fixed and is assumed as the real unit of a complex plane, with the directions of A_0C and its perpendicular as axes, then B_1 describes a circle with C as a center. Further, A_2 can move in any part of the plane within the range of the linkage and represents the product of the complex variables represented by B_1 and A_3, since $\Delta A_0CB_1 \sim \Delta A_0A_3A_2$ and $A_0C = 1$. In order to remove the restriction that B_1 and A_3 move on circles, in other words, to make the multiplication general, two linkages of the prescribed kind and with corresponding equal links may be pivoted together at their corresponding points A_0, C_1, B_1 and A_0', C_1', B_1'. Owing to the fact that in this compound linkage the triangles $A_0C_1B_1$ and $A_0'C_1'B_1'$ are identical, all the other corresponding triangles of this kind are similar. Thus, the remaining pivots of the second linkage being designated analogously by A_1', A_2', A_3', $\cdots$, two similar triangles $A_0A_2A_3$ and $A_0A_2'A_3'$ are obtained which otherwise are independent of each other.* Now fixing the points A_0, A_2 we take a_0a_2 as the first unit in the real axis of a new complex plane; then in every position of the compound linkage $\Delta A_0A_2A_3 \sim \Delta A_0A_2'A_3'$ and $A_0A_2 = 1$, so that A_3' represents the product of the complex variables represented by A_3 and A_2'. It is evident that by this linkage also the division of two complex numbers may be performed. Rotation and multiplication by a constant are special cases which may easily be arranged from the general linkage. Constructions for limiting positions of the linkage, which the reader may repeat without difficulty, show that the ranges of A_2', A_3' and A_3 are respectively within circles having A_0 as a center,

$$A_0D_1' + D_1'A_1' + A_1'D_2' + D_2'A_2' \qquad \text{and} \qquad A_0D' + D'A_3'$$

* On account of its complexity no separate drawing for the compound linkage has been made. A perfectly clear picture of it is obtained by keeping, in Fig. 4, A_0, C_1, B_1 fixed and giving the linkage a slight displacement. The original linkage and its displacement considered as one give the desired linkage.

as radii, A_2 as a center and $A_2D_3 + D_3A_3'$ as a radius. Fig. 4 is practically the same as the one given by KLEIBER in DYCK's catalogue. As in Fig. 1, which might be constructed with links of two different lengths, the effectiveness of KLEIBER's linkage is increased by taking $A_0D = DA_3 = A_3C = CA_0$, and also all other links equal to one another.

UNIVERSITY OF COLORADO, *March*, 1902.

NOTES AND ERRATA: VOLUMES 1, 2, 3

VOLUME 1

F. R. MOULTON: *On a class of particular solutions* ···.

P. 28, formula (26). *For* $\dfrac{x-\dfrac{1}{\sqrt{3}}}{\left\{\left(x-\dfrac{1}{\sqrt{3}}\right)\right\}^{\frac{3}{2}}}$ *read* $\dfrac{x-\dfrac{1}{\sqrt{3}}}{\left\{\left(x-\dfrac{1}{\sqrt{3}}\right)^2\right\}^{\frac{3}{2}}}$.

" l. 6 up. " 1.22 " 1.18 .

" l. " " −.24 " −.46 .

" l. 5 up. " −.93 " −.77 .

VOLUME 2

L. E. DICKSON: *Canonical forms of quaternary* ···.

P. 107, l. 1. *For* ξ' *read* ξ_1'.

P. 109, l. 7 up. " chose " choose.

P. 110, l. 4. " $L_{11}T_{1-1}, L_{2\mu}$ " $L_{11}T_{1-1}L_{2\mu}$.

P. 113, l. 22. " determines δ_{12} " determines β_{12}.

P. 121, l. 2 of § 15. " $=$ " $+$.

P. 125, middle. The number (33) refers only to the first of the two equations.

G. A. MILLER: *Determination of all the groups of order* p^m ···.

P. 263, l. 5. *For* t^7 *read* t_7.

" l. 10. " $p>3$ " $n=2,3,\cdots,p-2$.

" l. 11. *Read* $(t_7^{-1}t_6t_7)^{-1}P_2^n(t_7^{-1}t_6t_7)=P_3P_2^n=P_3^{n2}P_2^n=t_6^{-n}P_2^nt_6^n$.

P. 271, l. 4. *For* $0,\ p^4-1$ *read* $p^2-1,\ p^2(p^2-1)$.

Pp. 262, 263. From the second and third corrections it follows that the $p(p-1)$ subgroups of order p, mentioned in the second line from the bottom of p. 262, form two equal conjugate sets, and that the non-invariant subgroups of order p^2 and type $(1,1)$ contained in I are conjugate under I in sets of $(p-1)/2$ instead of forming a single conjugate set, as is stated in line 18 on p. 263. There are, therefore, eight groups of order $p^m(p>2)$ which are non-abelian and include the abelian group of type $(m-2,1)$; i. e., $p=3$

does not form an exception. The existence of G_8 when $p > 3$ may be proved in exactly the same manner as when $p = 3$. Each of the five groups G_1, G_3, G_5, G_7, G_8 is conformal with the abelian group of type $(m-3, 1, 1)$, G_2 and G_4 are conformal with the abelian group of type $(m-3, 2)$ while G_6 is conformal with the one of type $(m-1, 1)$. Four of these groups (G_1, G_2, G_5, G_6) contain invariant cyclic subgroups of order p^{m-2} while these subgroups are conjugate, in sets of p, in the remaining four groups.

W. F. OSGOOD: *On a fundamental theorem* ....

P. 278, l. 5. *After* point *insert* and no two curves corresponding to two distinct values of a will intersect each other.

E. J. WILCZYNSKI: *Geometry of a simultaneous system* ....

P. 359, l. 10 up. *For* form $y = \lambda\eta, z = \mu\zeta$ *read* form (2).

L. E. DICKSON: *Theory of linear groups in an arbitrary field.*

P. 370, l. 5. *For* $T_{s,-1} \cdots T_{3,-1}$ *read* $T_{2,-1} \cdots T_{s,-1}$.
P. 372, l. 4 up. *In* A'_{13}: $Y'_{12} = -Y'_{23}$, " Y_{23}.
P. 377, l. 15. *For* $\Sigma s'$ " $\Sigma s'$.
P. 384, l. 9. " $+ Y_{13}\eta_3$ " $+ Y_{12}\eta_3$.
P. 388, l. 15. " *subscript* $-\lambda\nu^{-1}$ " $-\lambda\nu$.
P. 388, l. 8 up. " $p^{6n}\Omega_1$ " $(p^{6n}-1)\,\Omega_1$.
P. 390, l. 7 up. " ξ_1 " η_1.
Pp. 383–391. For the simplicity of the group H' in the excluded case of modulus 2, see the report in the BULLETIN, November, 1902, of the Ninth Summer Meeting of the Society at Evanston.

VOLUME 3

O. STOLZ: *Zur Erklärung der Bogenlänge* ....

P. 31, l. 17. *For* $\sum_r f_r d_r$ *read* $\sum_r f_r \delta_r$.
P. 35, l. 13. " κ " Δ.

L. E. DICKSON: *The groups of Steiner in problems of contact.*

P. 44, l. 22. *For* $(00\, x_2y_2\, x_3x_3 \cdots)$ *read* $(00\, x_2y_2\, x_3y_3 \cdots)$.

C. N. HASKINS: *On the invariants* ....

P. 73, (2). *For* $\frac{\partial \xi_n}{\partial x_k}$ *read* $\frac{\partial \xi_r}{\partial x_k}$.

P. 75, (10). " $a_{ik}\ \underset{rl_1}{\xi_r}\ \underset{l_2}{}$ " $a_{ik}\ \underset{r}{\xi_r}\ \underset{l_1 l_2}{}$.

P. 77, (1). " $a_{kr}\ \underset{il_1 \cdots l_\mu l_{\mu+1}}{\xi_r}$ " $a_{kr}\ \underset{il_1 \cdots l_\mu l_{\mu+1}}{\xi_r}$.

P. 77, (2). " $a_{kr}\ \underset{il_1 \cdots l_\mu l_{\mu+1}}{\xi_r}$ " $a_{kr}\ \underset{il_1 \cdots l_\mu l_{\mu+1}}{\xi_r}$.

P. 80. In equation (1) the left member is $(\lambda\mu, \nu\rho)$.

P. 82, V. *For* $\underset{n}{\xi_n}$ *read* $\underset{n}{\xi_r}$.

P. 83, (9). " $\underset{n}{\xi_1}$ " $\underset{n}{0\xi_1}$.

P. 86. The numerator of the fraction in the last line is $|C_{11}C_{22}C_{33}|$.

P. 87. *For* $\begin{vmatrix} a_{12} & 2Z & 3Z_1 & 2Z_2 \\ a_{22} & 0 & z_2 & 0 \\ a_{11} & 0 & 0 & z_1 \\ a_{21} & 2Z & 2Z_1 & 3Z_2 \end{vmatrix}$ *read* $\begin{vmatrix} a_{12} & 2Z & 3Z_1 & 2Z_3 \\ a_{22} & 0 & Z_2 & 0 \\ a_{11} & 0 & 0 & Z_1 \\ a_{21} & 2Z & 2Z_1 & 3Z_2 \end{vmatrix}$.

E. B. VAN VLECK: *A determination of the number* ....

P. 130, l. 16. This line should be $E\left(\frac{X+2}{2}\right) + E\left(\frac{Y+2}{2}\right) - 2$.

P. 130, l. 17. *For* (22) *read* (21).

P. 130, l. 10 up. " $\lambda - \lambda'_i - \lambda'_j + 1$ " $\lambda'_1 + \lambda'_i > \lambda'_j + 1$.

E. H. MOORE: *On the projective axioms of geometry.*

Pp. 142–158. *In Hilbert's system* I, II *the axiom* I4 *is not redundant.* The error in my proof of the dependence of I4 lies in the omission of a citation of I4 in proof of the second statement of ll. 21–24, p. 143. That statement, call it I4*, is in effect: If two planes have their respective sets of incident points identical, they are themselves identical. The axiom I4 does depend upon I1–3, 5, 7, II 1–3, 5 *and* I4*.—E. H. M.

P. 143, l. 22. Insert the references (I2; I4).—The connection is in a 1-1 way.

P. 146, l. 6 up. *Omit* *indeed.*

P. 147, l. 4. *For* *thus* *read* *hence.*

P. 155, l. 13. " segment " line.

" α " α'.

P. 156, l. 15. " k-space " $(k-1)$-space.